A Gentle Introduction to the Edgeworth Box and Pareto Efficiency

Maria Betto

December 2, 2021

Introduction

The Edgeworth box is a useful tool for visualizing simple pure exchange economies. The setting has the following ingredients:

- Two agents, 1 and 2.
- Two goods, x and y.
- Agent 1 and 2's preferences for x and y, described by the utility functions u_1 and u_2, respectively.
- Agent 1 and 2's initial endowments of goods x and y.

This simple setting allows for a nice and convenient two-dimensional graphical representation.

Constructing the box

We begin with an example. Suppose Ann (1) and Bob (2) are the only two people in a pure exchange economy. There are only two goods in this economy, x and y. Ann starts with 7 units of x and 3 units of y. Bob starts with 3 units of x and 3 units of y.

In total, there are 10 units of x and 6 units of y to go around.

In symbols, we typically use e to denote endowments, or the quantities people start with. That is, we will write $e^{\square}_{\triangle}$ to denote the quantity of good $\square$ (here, either x or y) person $\triangle$ (here, either 1 or 2) starts with. Similarly, $e^{\square}$ (i.e. when the subscript is omitted) is used to denote the total amount of good $\square$ available in this economy.

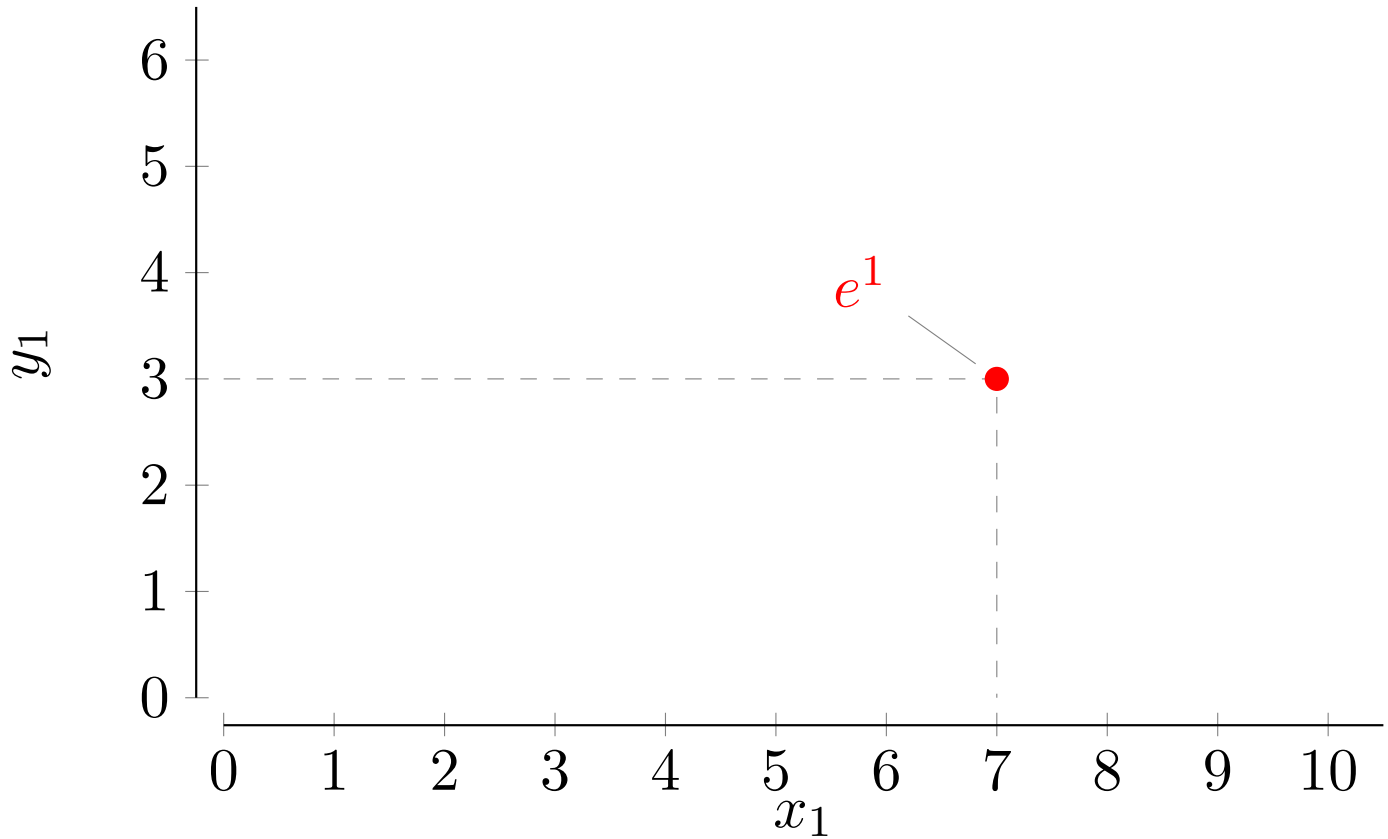
In our example, we have that:

$$e_1^x = 7 \qquad e_1^y = 3$$
$$e_2^x = 3 \qquad e_2^y = 3$$

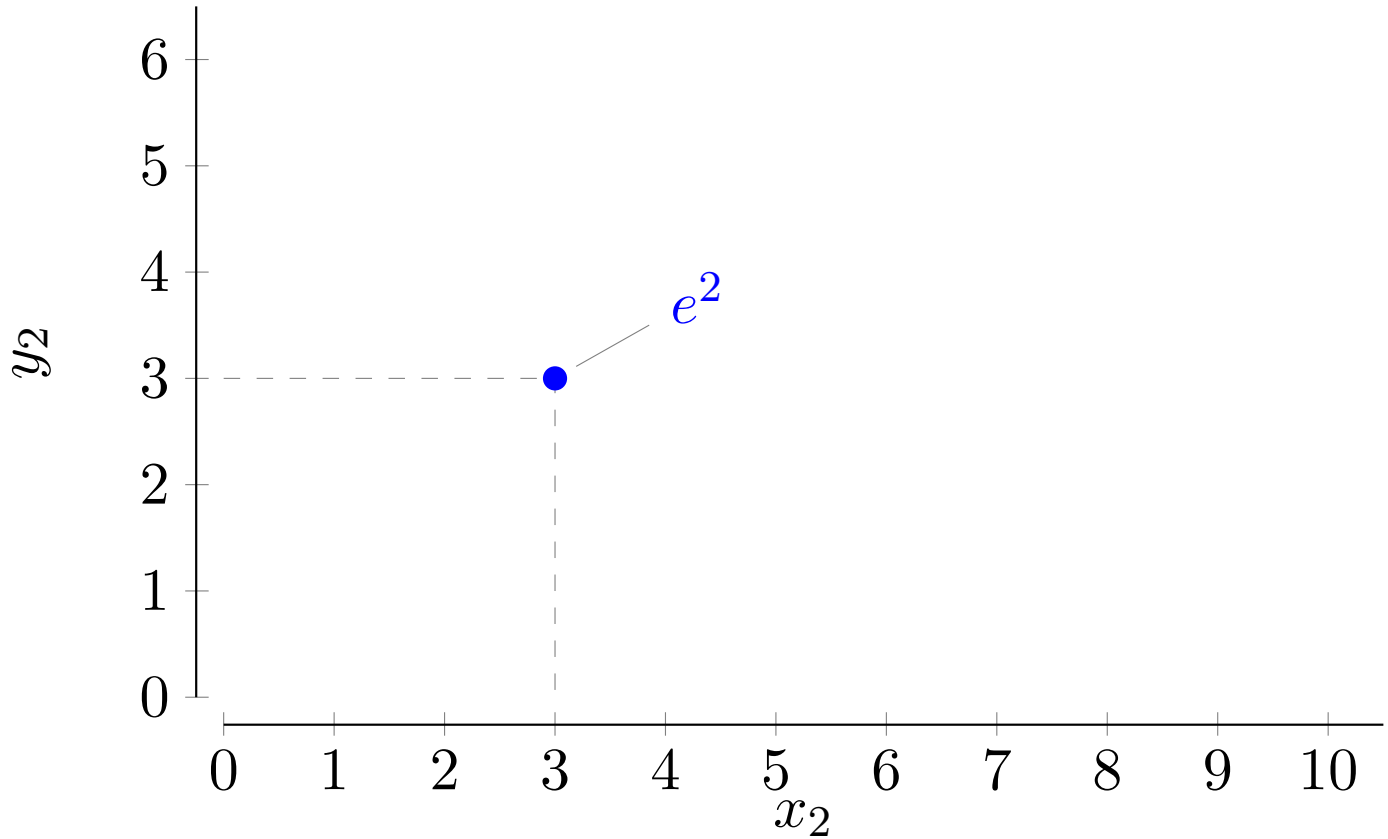
and thus,

$$e^x = 10 \qquad e^y = 6.$$

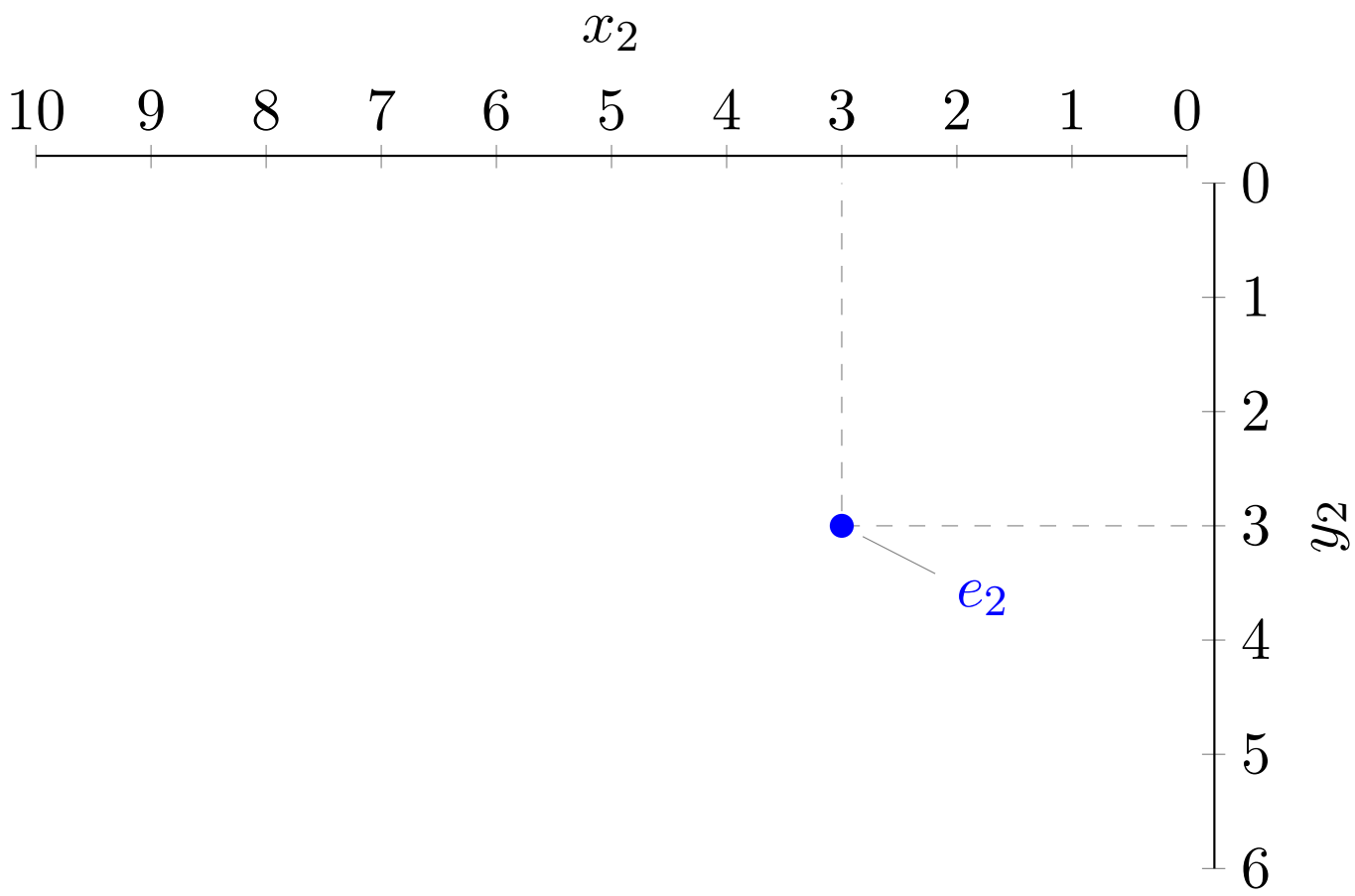
From Ann's (1) perspective, we can plot her endowment point in a diagram. The width of the diagram is $e^x = 10$ and the height is $e^y = 6$. Ann's endowment e^1 is then the point $(7, 3)$ on this diagram:



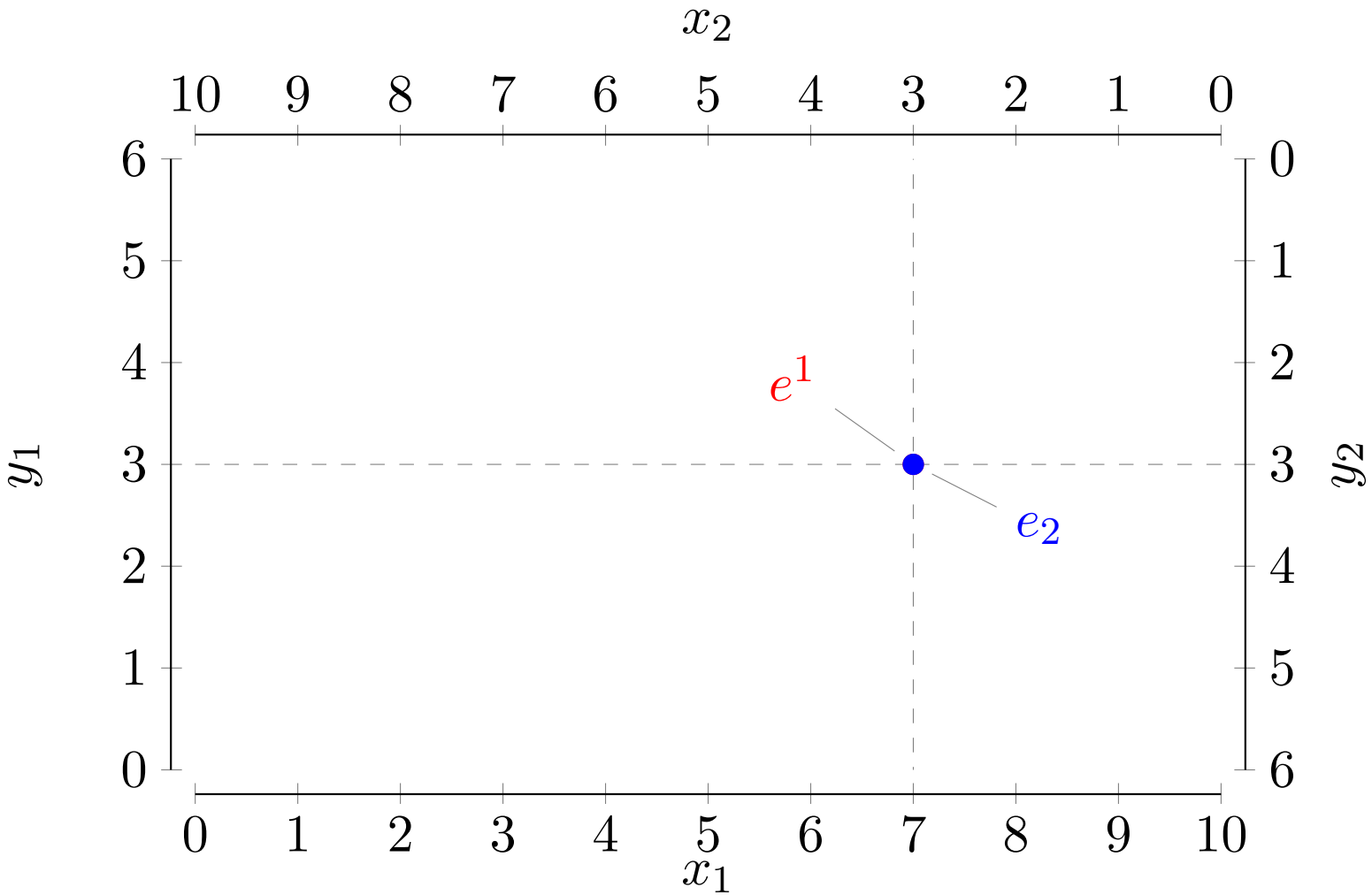
Of course, we could do the same from Bob's (2) perspective:



Let us flip Bob's (2) diagram ...

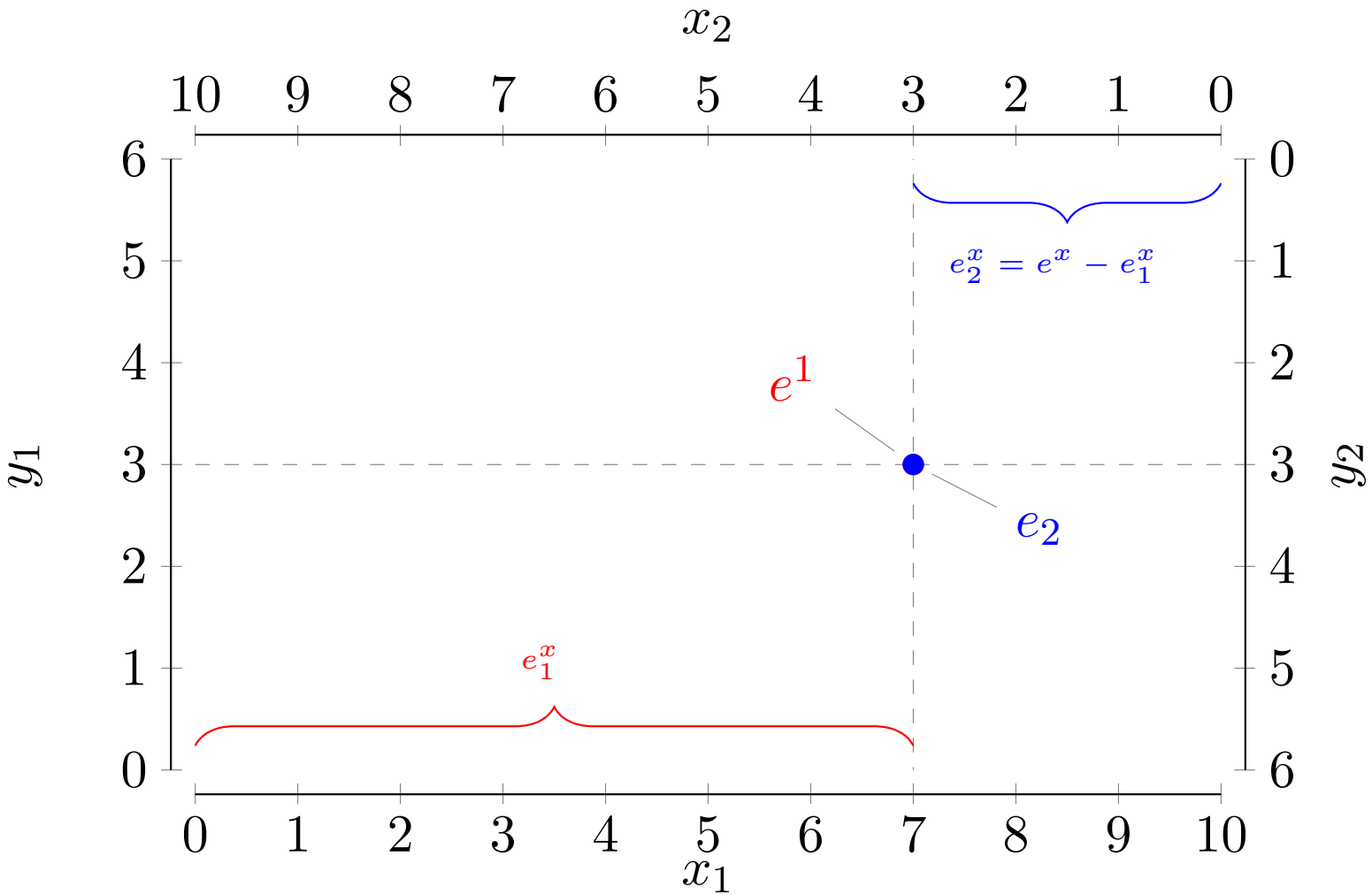


... and superimpose it with Ann's (1):

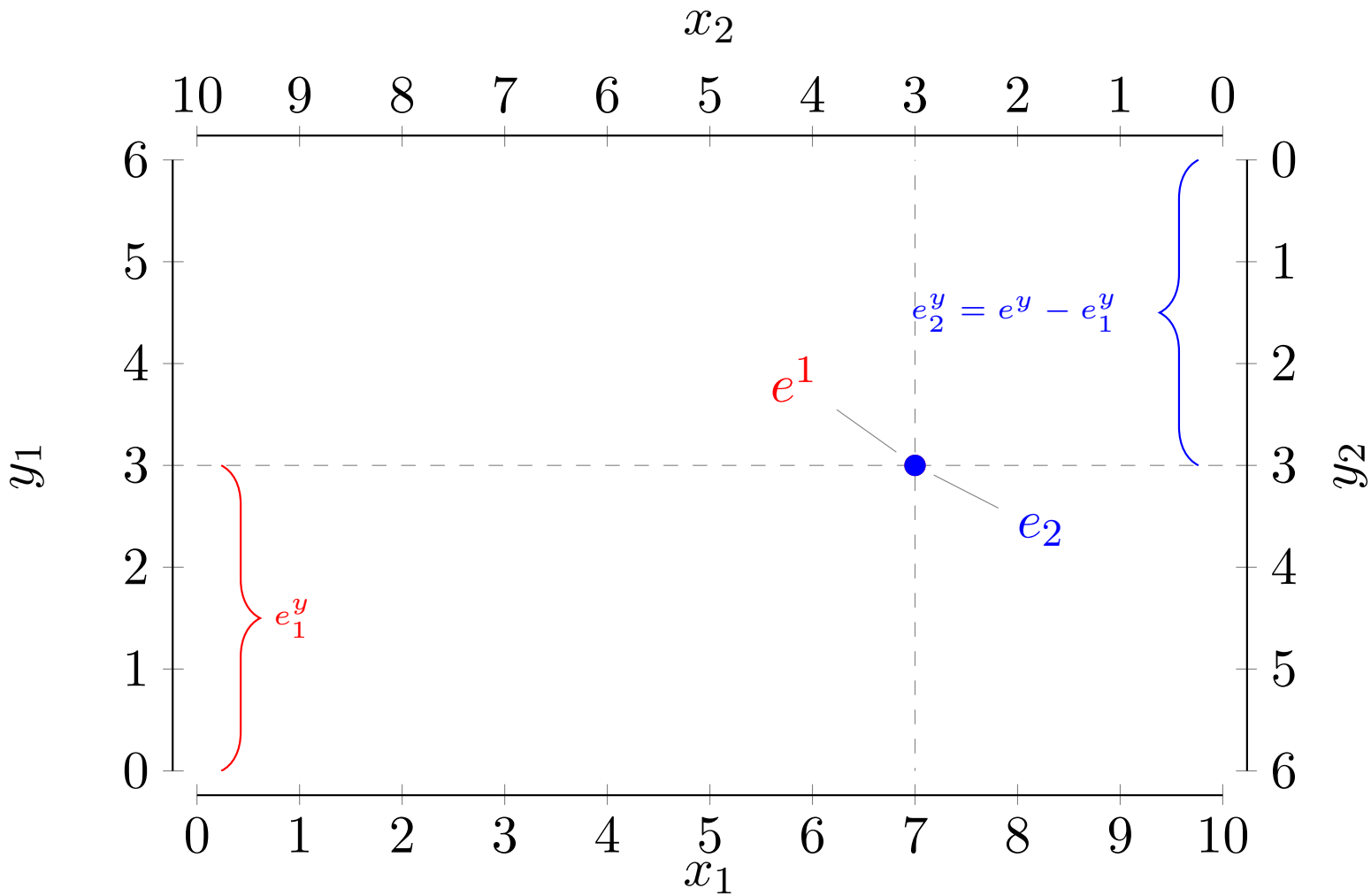


Geometrically, e_1 and e_2 are both in the same place (what we will call the *endowment point*). This is not a coincidence, but a product of our construction, since...

Box width $= e^x = e_1^x + e_2^x$



$$\text{Box height } = e^y = e_1^y + e_2^y$$

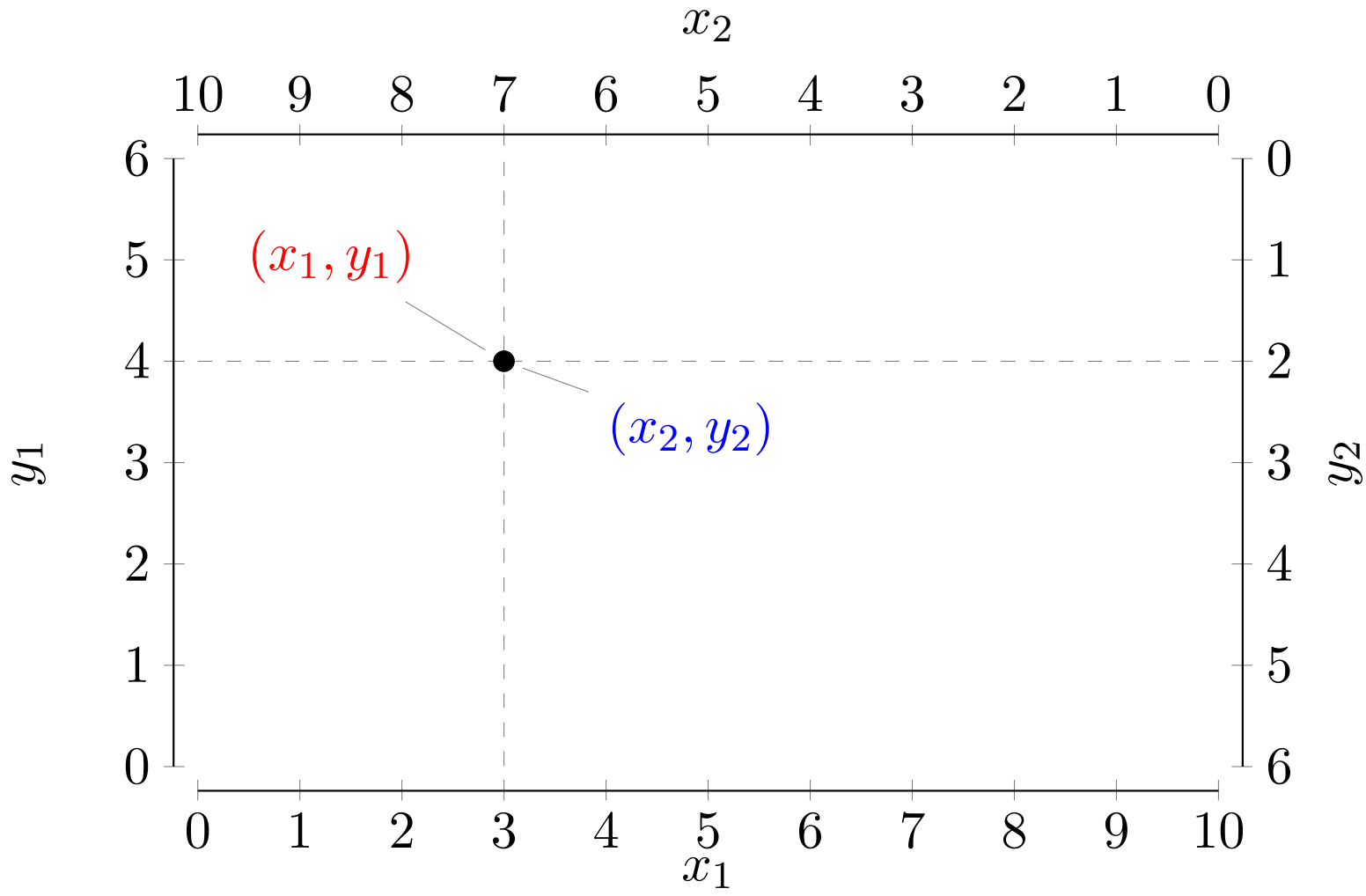


Feasibility

A **feasible allocation** (x_1, y_1), (x_2, y_2) is one that exactly exhausts all of the available resources in this economy, such that there is no waste. That is, (x_1, y_1), (x_2, y_2) is *feasible* whenever:

$$x_1 + x_2 = e_1^x + e_2^x = e^x = 10$$
$$y_1 + y_2 = e_1^y + e_2^y = e^y = 6.$$

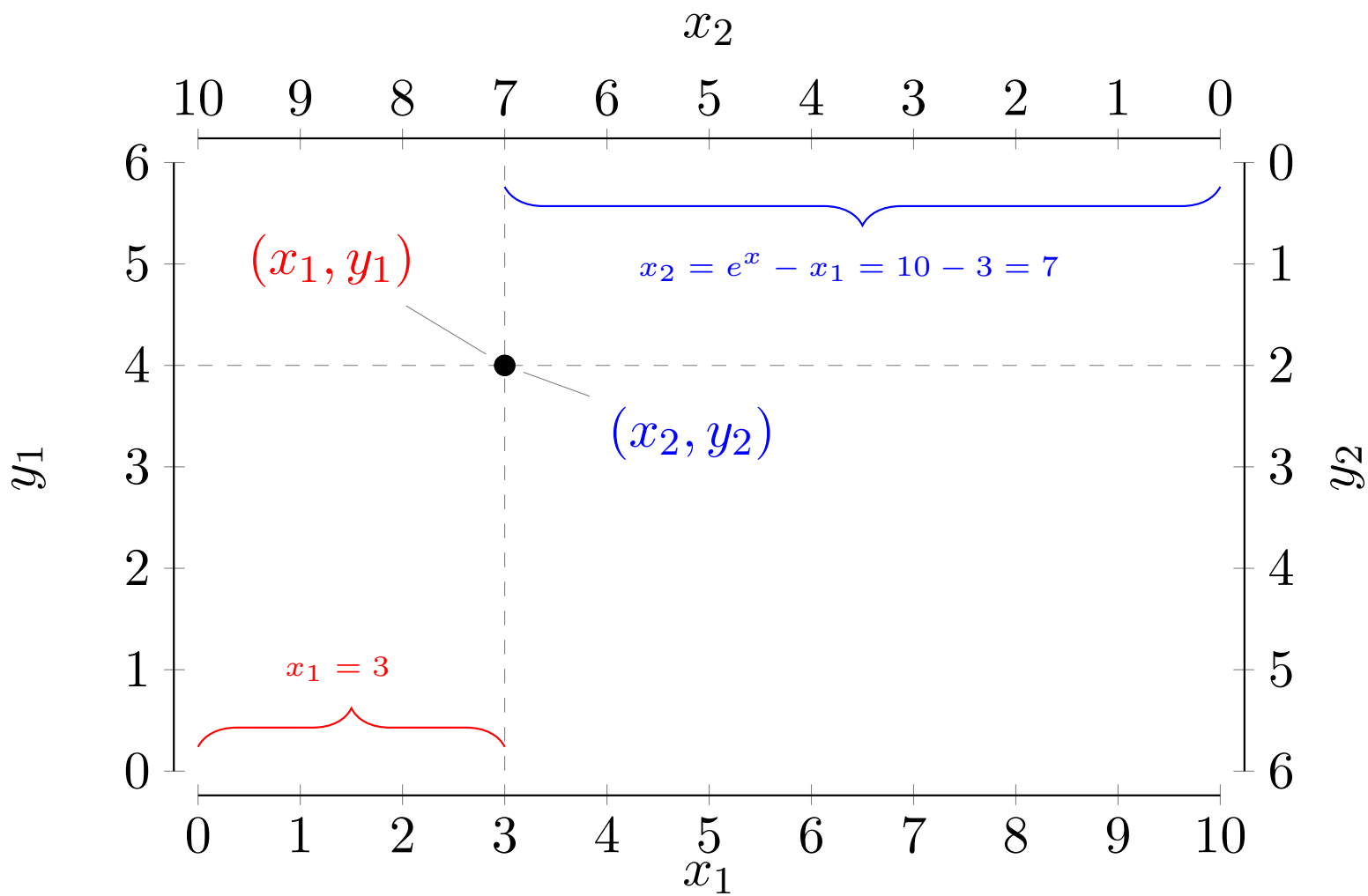
Any point inside the box corresponds to a feasible allocation. Indeed, any point inside the box has two sets of coordinates – Ann's (1, right side up) or Bob's (2, upside down) – which add up to the dimensions of the box. For example ...



... $(x_1, y_1) = (3, 4)$ or $(x_2, y_2) = (7, 2)$

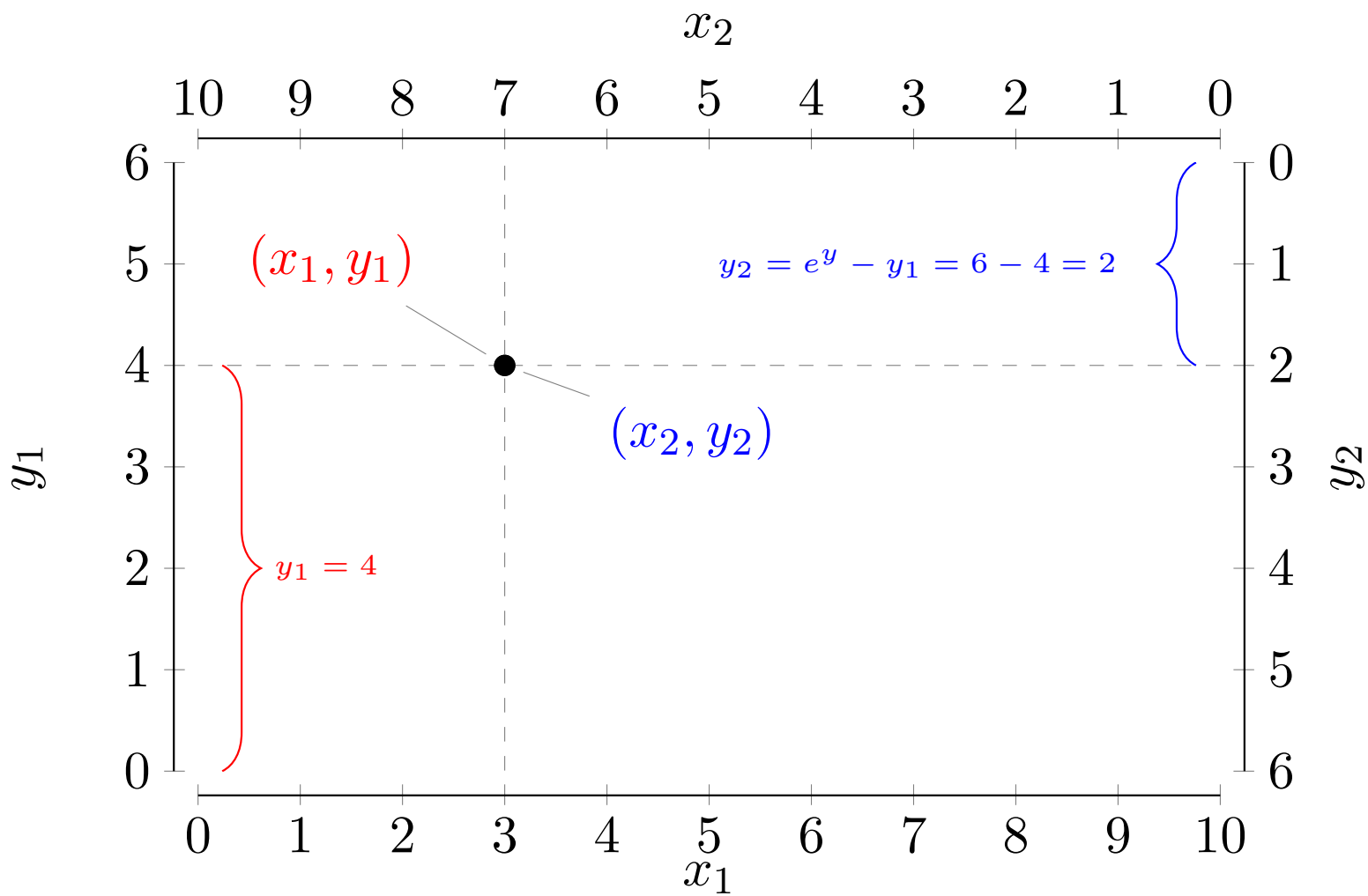
$$x_1 + x_2 = e^x = \text{ Box width}$$

$$3 + 7 = 10$$



$$y_1 + y_2 = e^y = \text{ Box height}$$

$$4 + 2 = 6$$

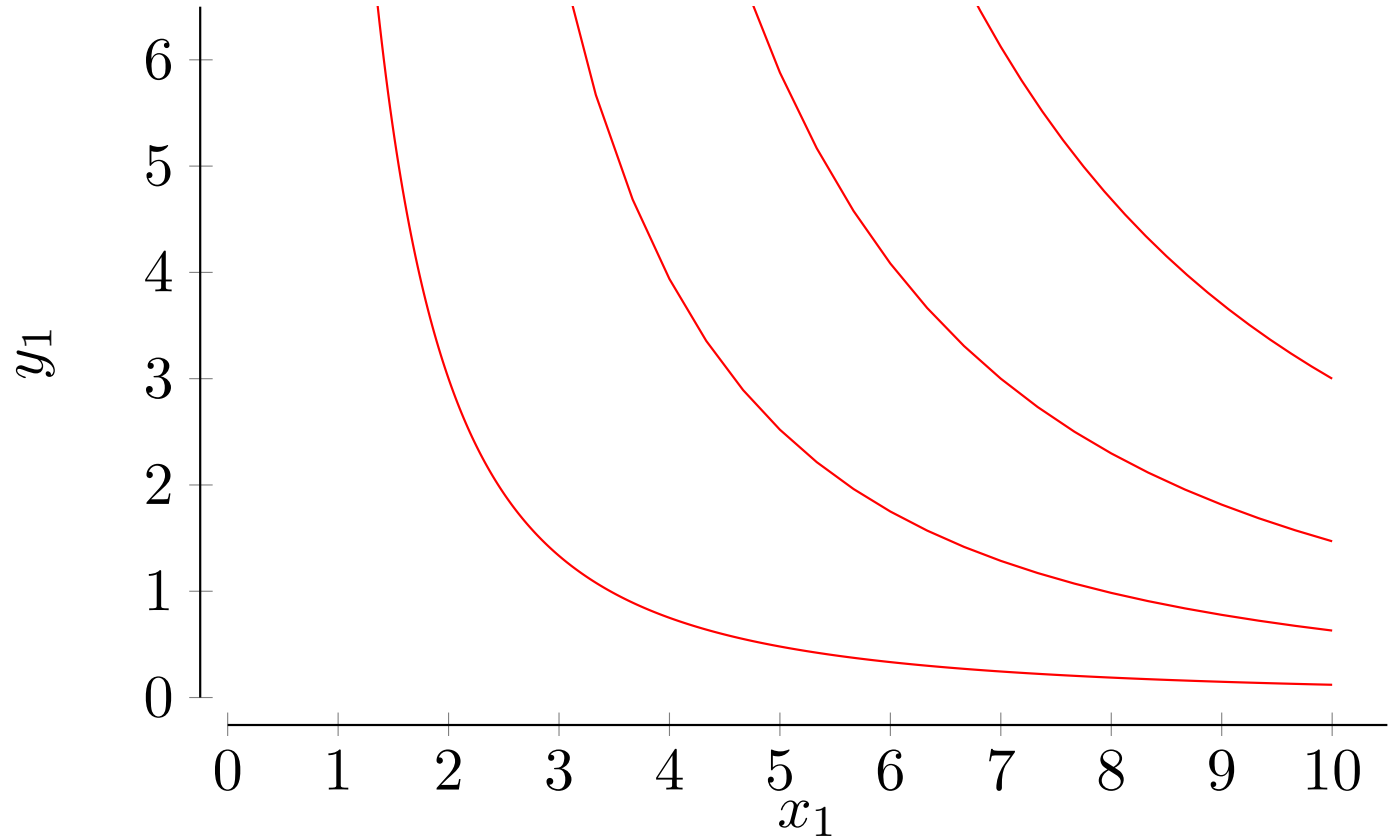


Preferences and Pareto dominance

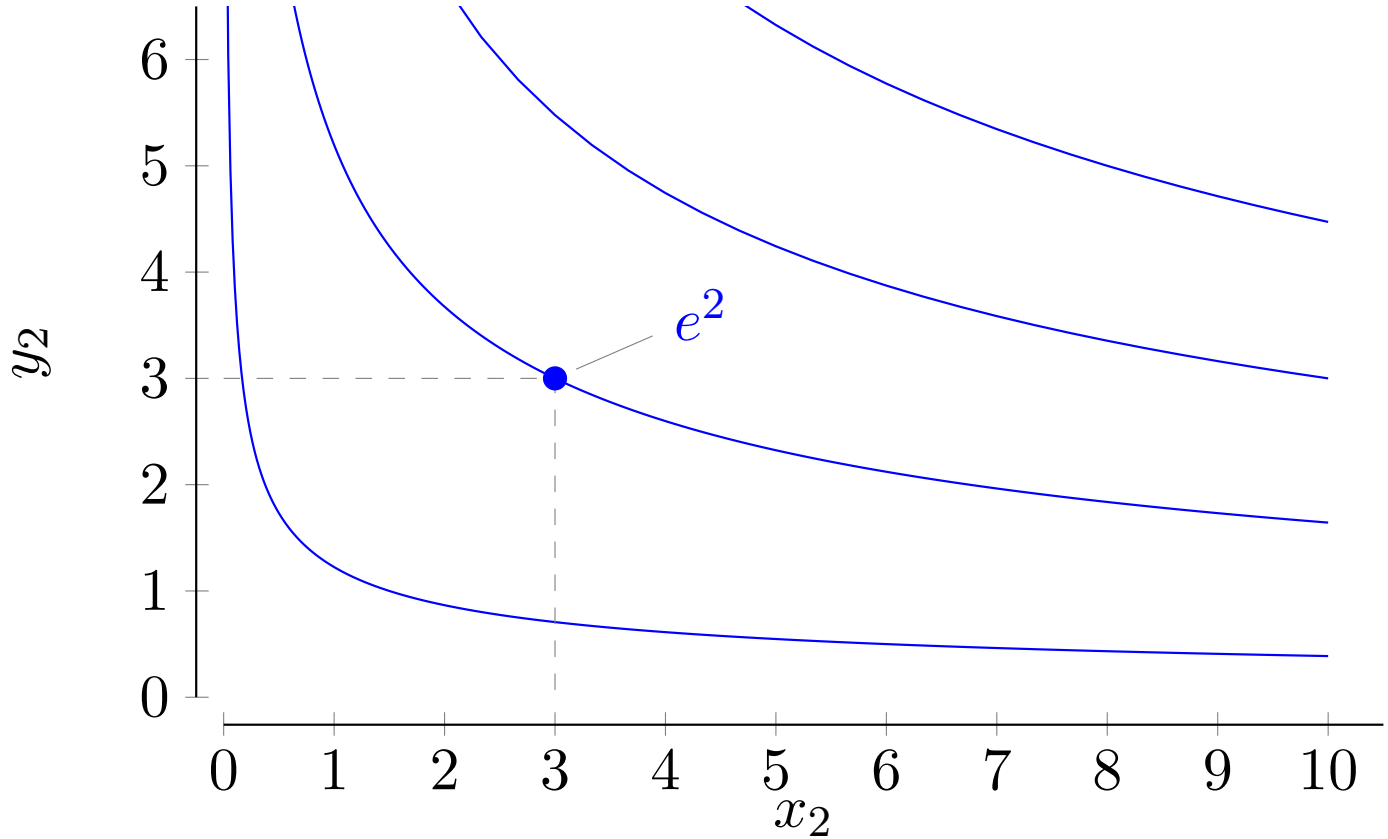
Ann and Bob have preferences over the different feasible allocations. For example, let

$$u_1(x_1, y_1) = x_1^2 y_1$$
$$u_2(x_2, y_2) = x_2 y_2^2.$$

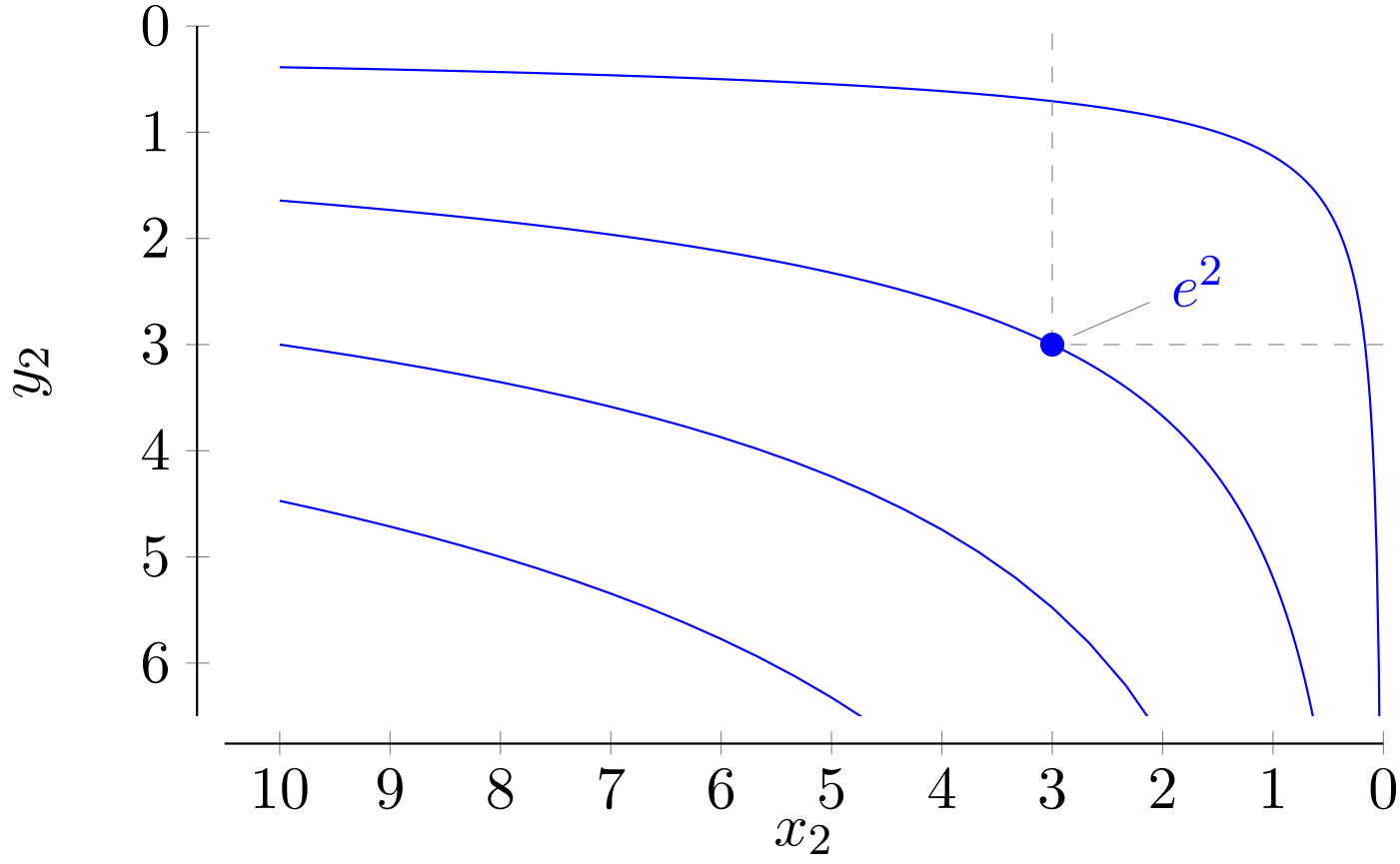
Let's draw a few indifference curves for Ann (1).


y_1
6
5
4
3
2
1
0
0
1
2
3
4
5
6
7
8
9
10
x_1

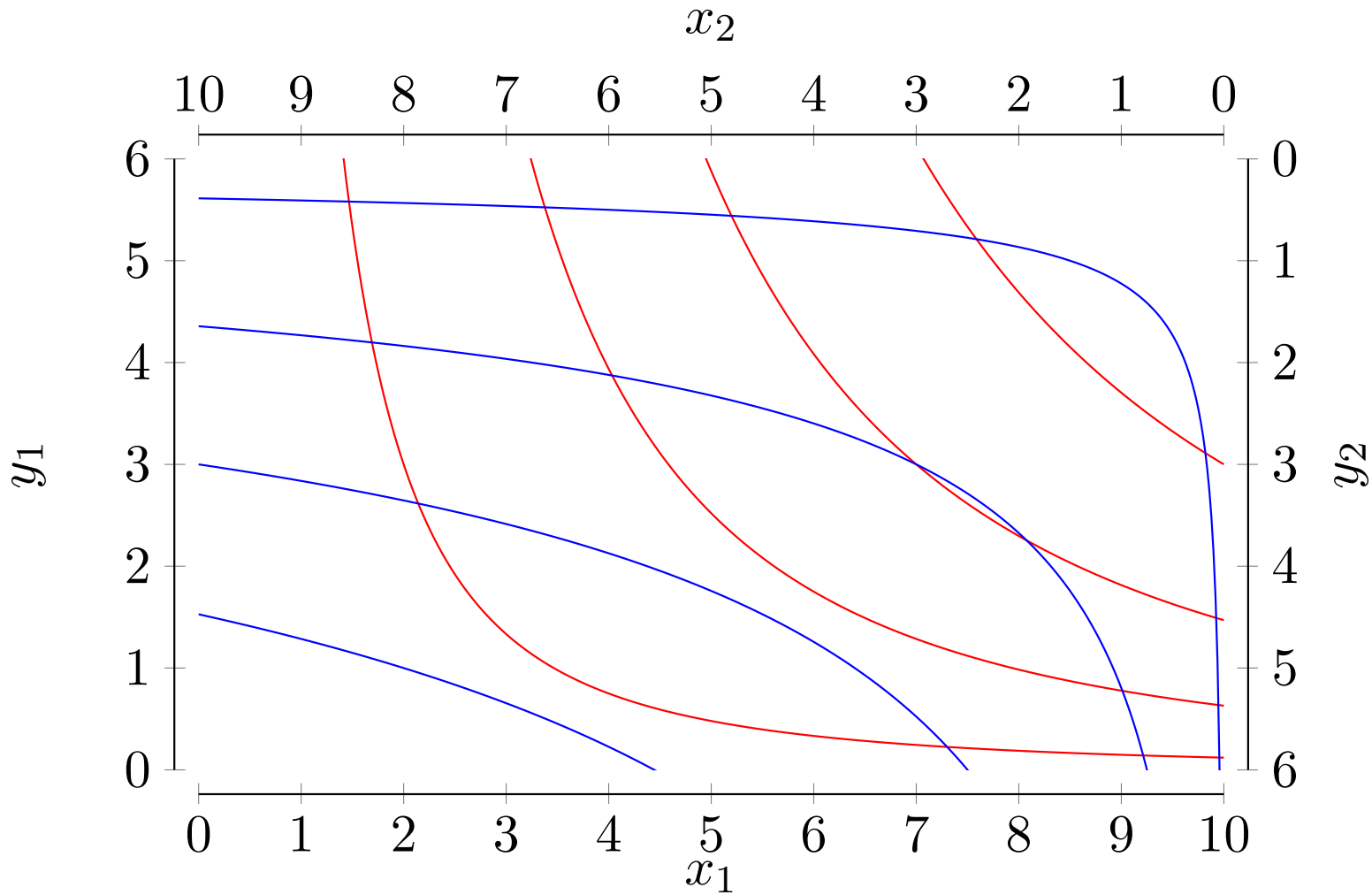
We'll do the same for Bob (2).



Let us flip Bob's (2) diagram.



And superimpose it with Ann's (1):

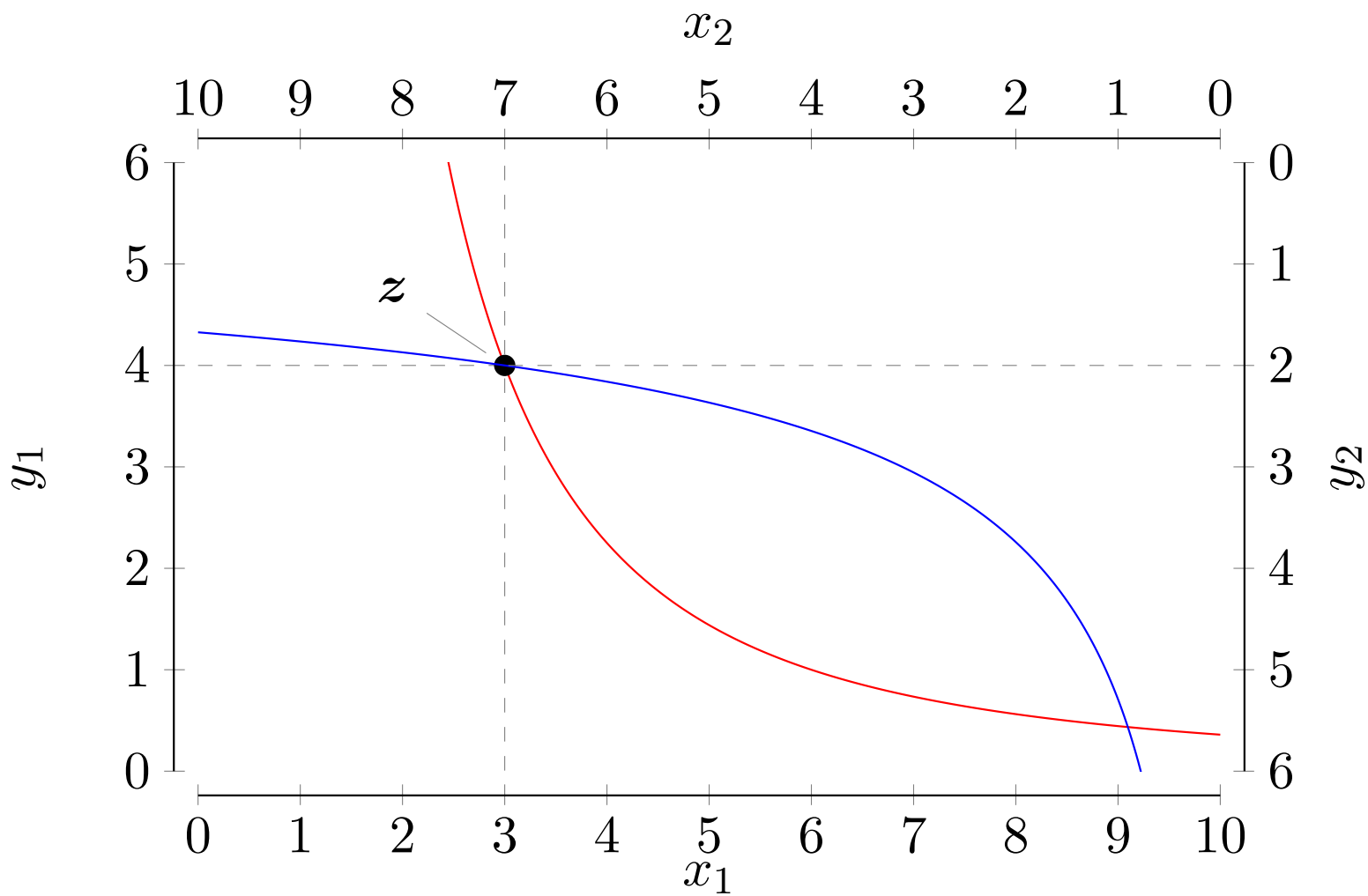


Pareto superior

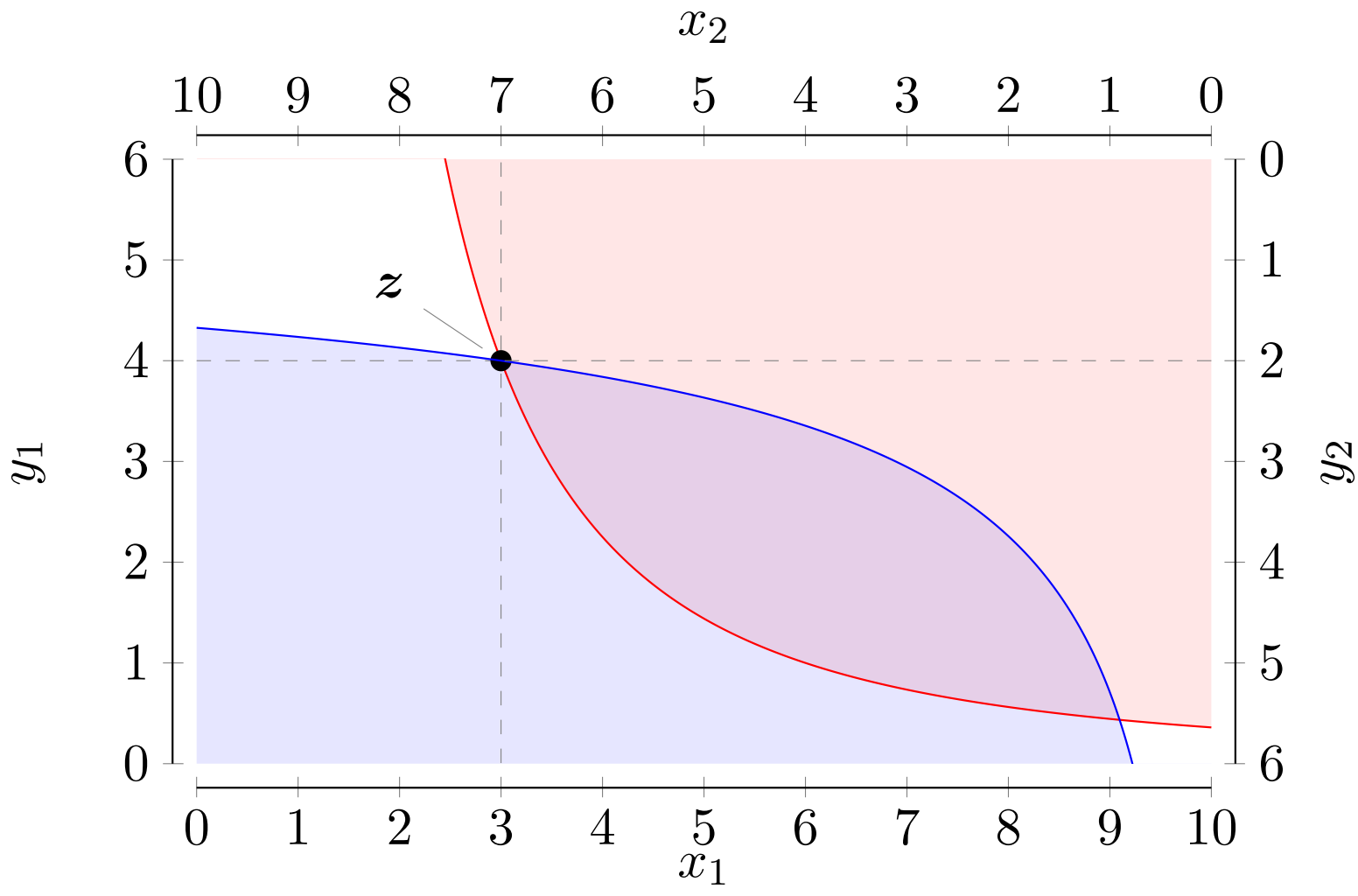
An allocation $\boldsymbol{z}$ is *Pareto superior* to (or Pareto dominates) an allocation $\boldsymbol{z}'$ if every consumer prefers $\boldsymbol{z}$ to $\boldsymbol{z}'$, and at least one consumer strictly prefers $\boldsymbol{z}$ to $\boldsymbol{z}'$.

We can equivalently say that $\boldsymbol{z}'$ is *Pareto inferior* to $\boldsymbol{z}$ (or is Pareto dominated by $\boldsymbol{z}$).

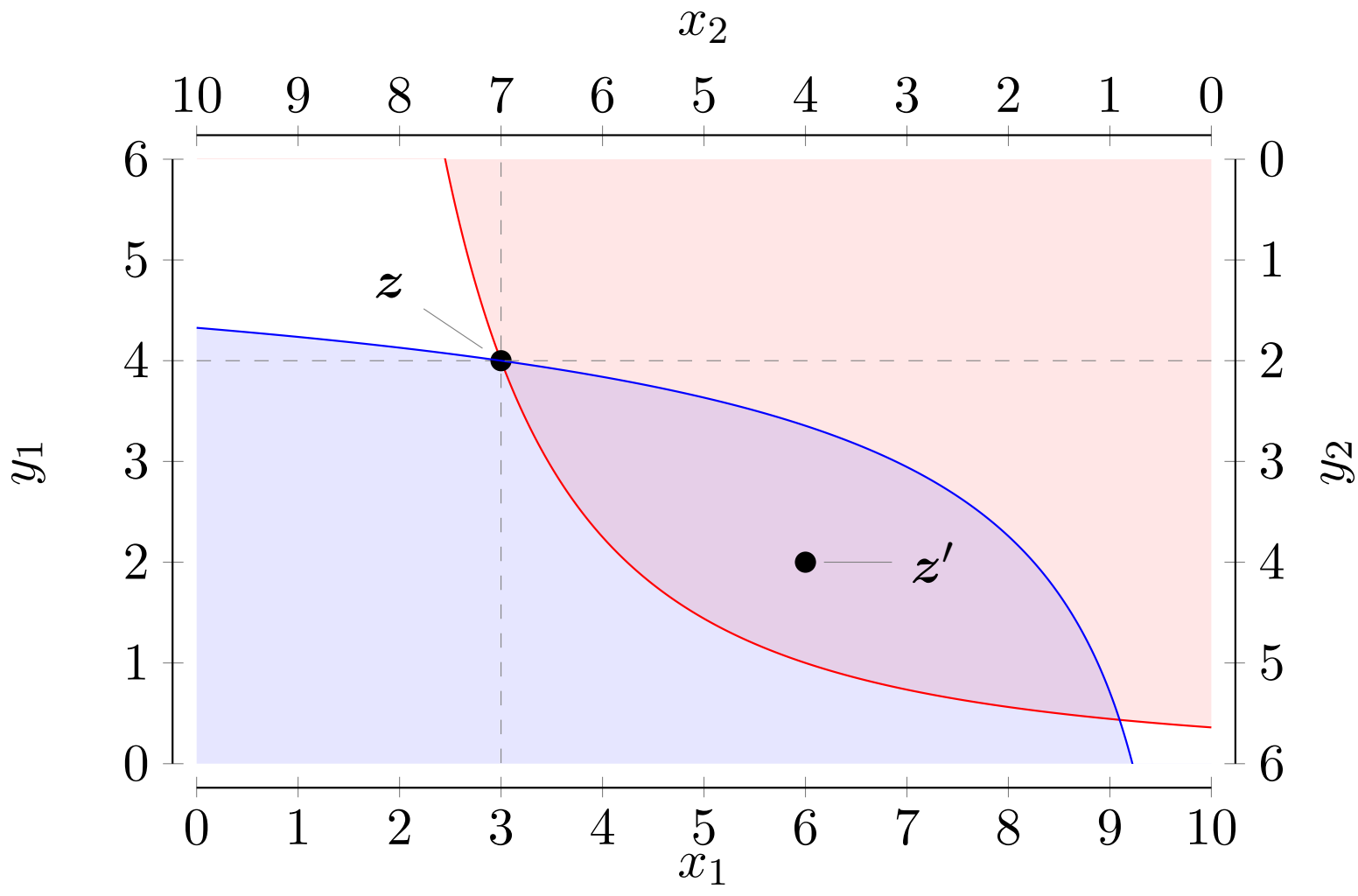
As an example, let's analyze point $\boldsymbol{z} := (x_1, y_1) = (3, 4)$ (or, equivalently, $(x_2, y_2) = (7, 2)$).



Given the indifference curves that go through $\boldsymbol{z}$, we are able to determine which points Ann (1, red) likes better than $\boldsymbol{z}$, and which points Bob (2, blue) likes better than $\boldsymbol{z}$.


x_2
10 9 8 7 6 5 4 3 2 1 0
y_1
6 5 4 3 2 1 0
z
y_2
0 1 2 3 4 5 6
0 1 2 3 4 5 6 7 8 9 10
x_1

Any point in the doubly-shaded lens-shaped region is preferable to *both* Ann and Bob. For example, $\boldsymbol{z}' := (x_1, y_1) = (6, 2)$ (or, equivalently, $(x_2, y_2) = (4, 4)$).


x_2
10 9 8 7 6 5 4 3 2 1 0
y_1
0 1 2 3 4 5 6
y_2
0 1 2 3 4 5 6
z
z'
0 1 2 3 4 5 6 7 8 9 10
x_1

We say that $\boldsymbol{z}'$ is *Pareto superior*, or Pareto dominates $\boldsymbol{z}$.
Equivalently, $\boldsymbol{z}$ is *Pareto inferior* to, or Pareto dominated by $\boldsymbol{z}'$.

Are there any allocations $\boldsymbol{z}$ that aren't Pareto inferior to any other allocations $\boldsymbol{z}'$?

Pareto efficient

An allocation $\boldsymbol{z}$ is said to be *Pareto efficient* or *Pareto optimal* if no other feasible allocation $\boldsymbol{z}'$ is Pareto superior to $\boldsymbol{z}$ (or, equivalently, if $\boldsymbol{z}$ is not Pareto inferior to, or dominated by, another feasible allocation $\boldsymbol{z}'$).

In our example, the idea is to find points that do not generate a "lens-shaped" area between two indifference curves. This is accomplished by finding the points where Ann (1) and Bob's (2) indifference curves are tangent to each other. Recall:

$$u_1(x_1, y_1) = x_1^2 y_1$$
$$u_2(x_2, y_2) = x_2 y_2^2.$$

Then,

$$MRS_1^{x,y}(x_1, y_1) = \frac{MU_1^x}{MU_1^y} = \frac{2y_1}{x_1}$$
$$MRS_2^{x,y}(x_2, y_2) = \frac{MU_2^x}{MU_2^y} = \frac{y_2}{2x_2}.$$

Recall that the slope of Ann's (1) indifference curve that goes through the point (x_1, y_1) is $-MRS_1^{x,y}(x_1, y_1)$. Similarly, the slope of Bob's (2) indifference curve that goes through the point (x_2, y_2) is $-MRS_2^{x,y}(x_2, y_2)$.

To find the points of tangency between Ann and Bob's indifference curves, we'd like to set the two marginal rates of substitution equal to each other. We can't do that right away because they were expressed in different sets coordinates: Ann's was expressed in terms of x_1, y_1 and Bob's, x_2, y_2.

Luckily, we can use feasibility to translate them into the same set of coordinates. All we need to do is use that $x_1 + x_2 = e^x = 10$ and $y_1 + y_2 = e^y = 6$ at any feasible points (x_1, y_1),(x_2, y_2). We substitute for $x_2 = e^x - x_1 = 10 - x_1$ and $y_2 = e^y - y_1 = 6 - y_1$ and obtain:

$$\begin{aligned}
MRS_1^{x,y} &= MRS_2^{x,y} \\
\iff \quad & \frac{2y_1}{x_1} = \frac{y_2}{2x_2} \\
\iff \quad & \frac{2y_1}{x_1} = \frac{6 - y_1}{2(10 - x_1)} \\
\iff \quad & 40y_1 - 4x_1y_1 = 6x_1 - x_1y_1 \\
\iff \quad & y_1 = \frac{6x_1}{40 - 3x_1}.
\end{aligned}$$

for all positive values of x_1, x_2, y_1, y_2.

The expression we've just obtained,

$$y_1 = \frac{6x_1}{40 - 3x_1},$$

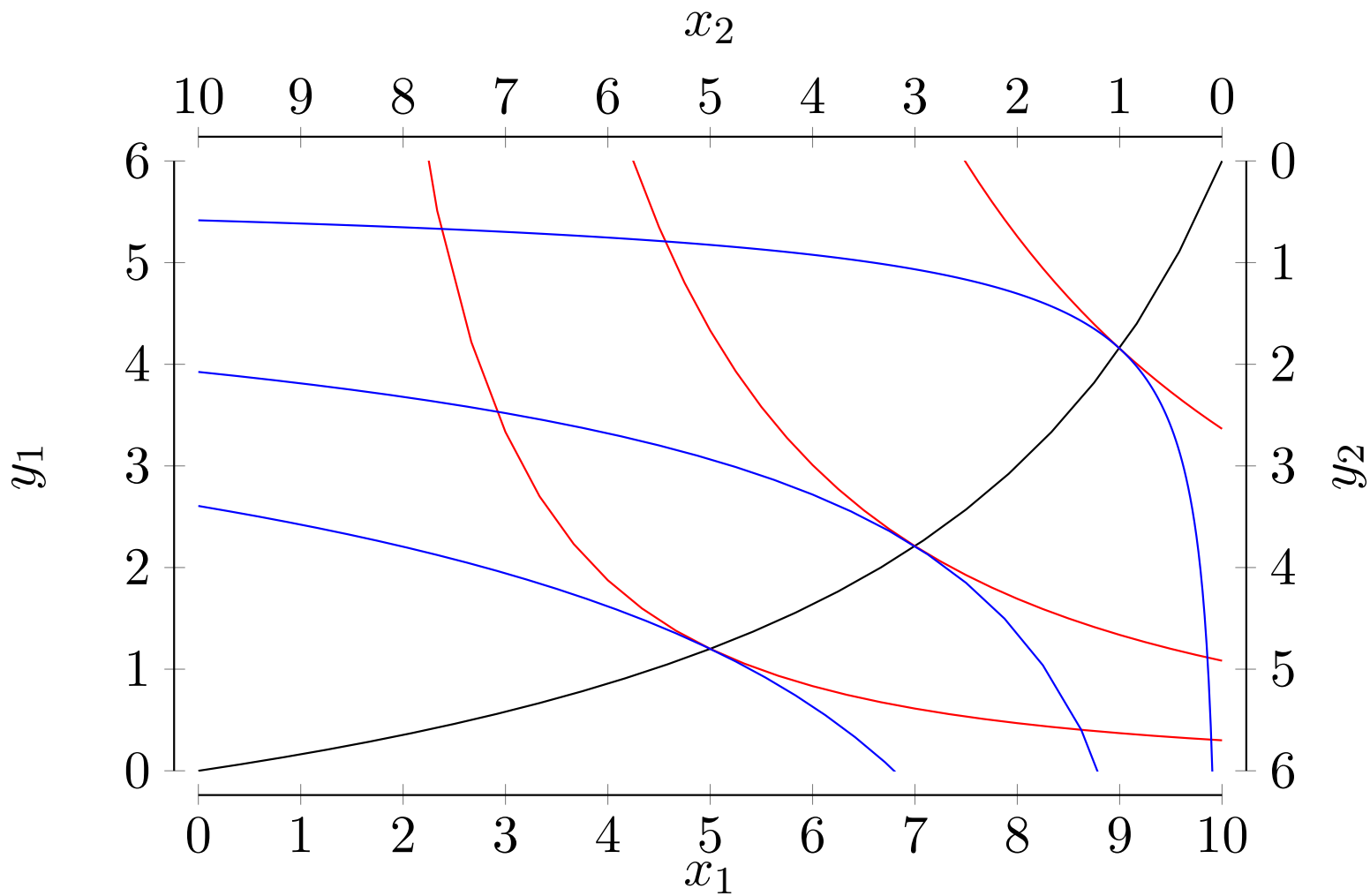
has the location of all *interior* allocations where Ann and Bob's indifference curves are tangent to each other, expressed in x_1, y_1 coordinates. Note that we could easily express the same curve in x_2, y_2 coordinates (by using feasibility):

$$6 - y_2 = \frac{6(10 - x_2)}{40 - 3(10 - x_2)}$$

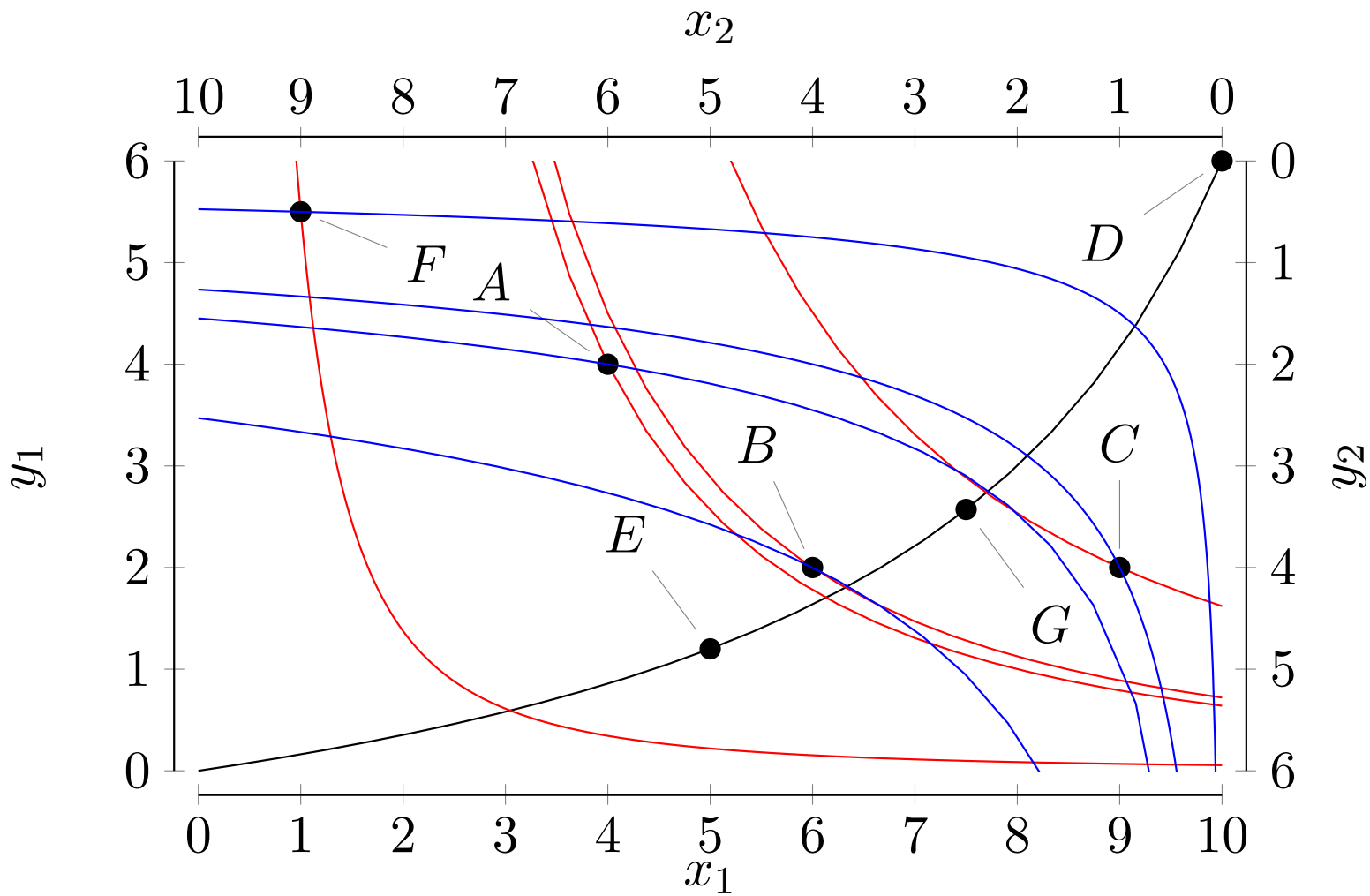
$$\implies y_2 = \frac{24x_2}{10 + 3x_2}.$$

These two expressions naturally describe the exact same curve in the Edgeworth box.

We plotted in black the curve identified in the previous slide. By construction, Ann and Bob's indifference curves are tangent anywhere along this curve.



Exercise. Which of the allocations A, B, C, D, E, F or G below are Pareto efficient? Which allocations Pareto dominate A? Which allocations are Pareto dominated by A?



Answer. D, E and G are Pareto efficient. B and G Pareto dominates A, and A Pareto dominates F.

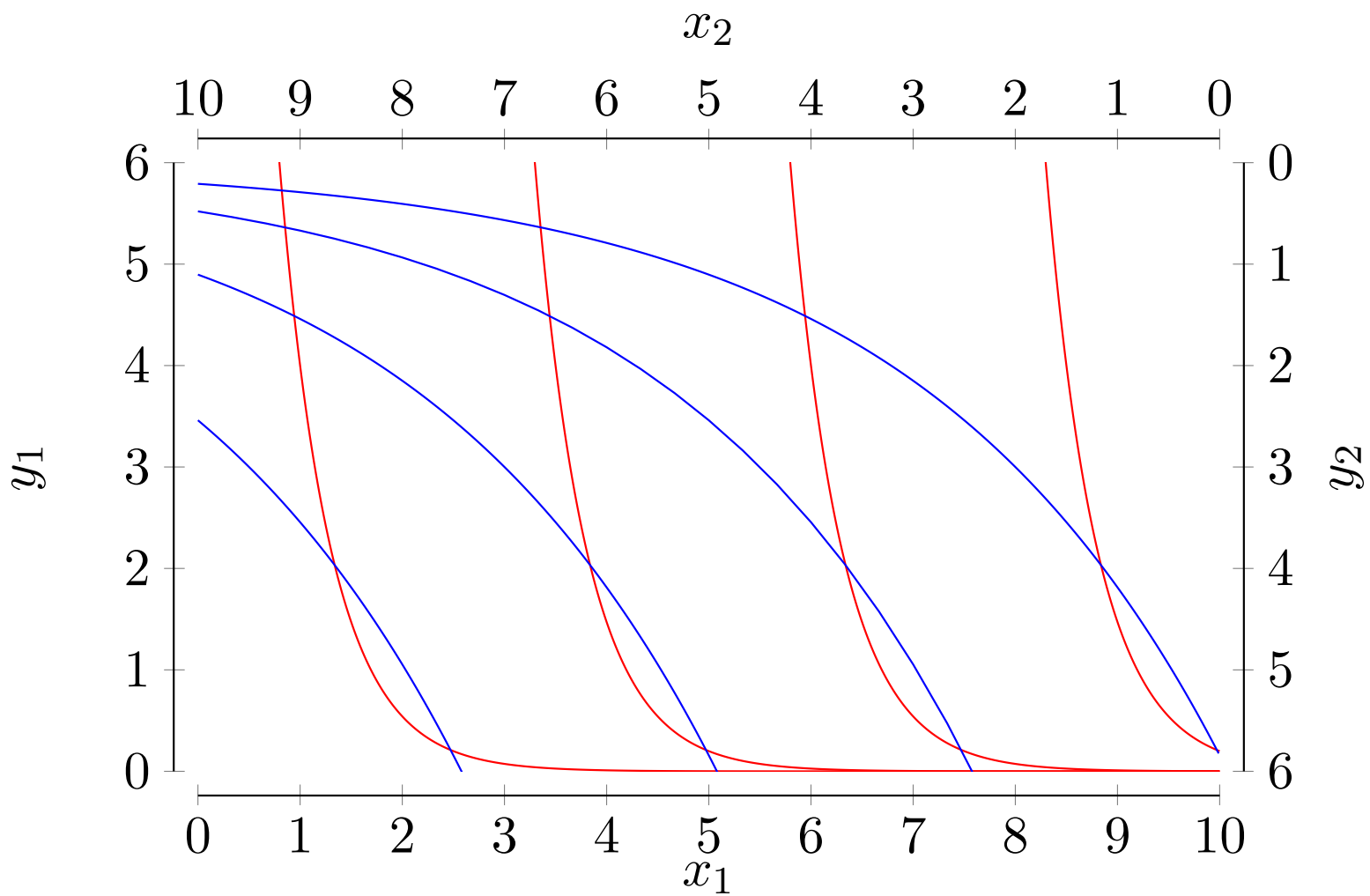
Note that while D and E are Pareto efficient, they do not Pareto dominate A. Also note that while B Pareto dominates A, it is not Pareto efficient.

Examples

Let's take a look at different examples. We will keep $e^x = 10$ and $e^y = 6$.

$$u_1(x_1, y_1) = 2x + \ln(y)$$
$$u_2(x_2, y_2) = x + 3\ln(y)$$

This is what the indifference curves look like in the box.



Preferences that can be expressed as:

$$u(x, y) = bx + v(y),$$

where v is an increasing function, are called **quasi-linear** (in y), and have the interesting property that their marginal rates of substitution depend only on y (the quasi-linear good):

$$MRS^{x,y}(x, y) = \frac{b}{v'(y)}$$

Since the MRS is the absolute value of the slope of the indifference curves, this implies that all indifference curves of quasi-linear (in y) preferences are parallel translations of each other along the horizontal axis.

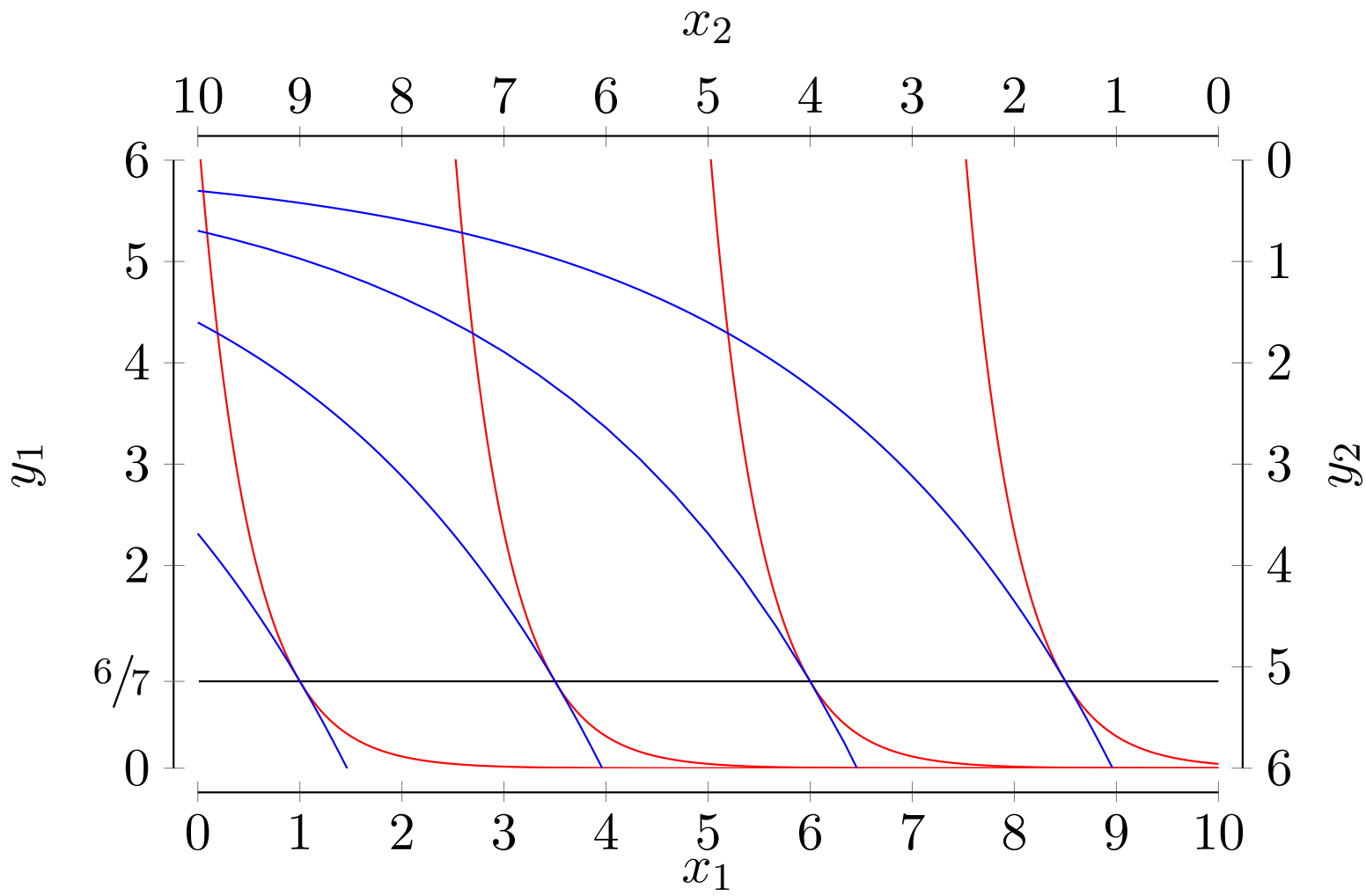
Let's characterize the set of all Pareto efficient allocations.

We begin by computing the two marginal rates of substitution, as before.

$$MRS_1^{x,y} = \frac{MU_1^x}{MU_1^y} = \frac{2}{1/y_1} = 2y_1,$$
$$MRS_2^{x,y} = \frac{MU_2^x}{MU_2^y} = \frac{1}{3/y_2} = \frac{y_2}{3}.$$

Using feasibility, i.e. that $y_2 = 6 - y_1$, and equating the MRS's yields:

$$2y_1 = \frac{6 - y_1}{3} \implies y_1 = \frac{6}{7}.$$


x_2
10 9 8 7 6 5 4 3 2 1 0
6 5 4 3 2 6/7 0
y_1
0 1 2 3 4 5 6
y_2
0 1 2 3 4 5 6 7 8 9 10
x_1

We are not yet done. We have found all the *interior Pareto efficient allocations*, obtained by setting $MRS_1^{x,y} = MRS_2^{x,y}$. But what about the boundaries?

Boundaries arise due to *nonnegativity constraints* – that is, the imposition that both individuals must consume nonnegative amounts of x and y. This extra constraint on the variables sometimes binds, either because it prevents the point where $MRS_1^{x,y} = MRS_2^{x,y}$ from being reached, or simply because $MRS_1^{x,y} = MRS_2^{x,y}$ never holds.

The reason why we must analyze the boundaries separately is as follows. Suppose $MRS_1^{x,y} > MRS_2^{x,y}$ at an allocation $\boldsymbol{z}$. That is, Ann's (1) enjoyment of x relative to y is greater than Bob's (2). Then, there's typically a better allocation (aka Pareto superior to $\boldsymbol{z}$) where Ann (1) gives up some of her y to Bob (2) in exchange for some of his x. That would imply $\boldsymbol{z}$ is not Pareto efficient, unless of course such Pareto-improving trades can't happen.

Then, if $MRS_1^{x,y} > MRS_2^{x,y}$ at an allocation $\boldsymbol{z}$, but either $x_1 = e^x$ or $y_1 = 0$, then $\boldsymbol{z}$ must be Pareto efficient, as the Pareto-improving trades are disallowed due to nonnegativity constraints. Similarly, if $MRS_1^{x,y} < MRS_2^{x,y}$ at $\boldsymbol{z}$ and either $x_1 = 0$ or $y_1 = e^y$, nonnegativity of variables implies $\boldsymbol{z}$ must be Pareto efficient.

Let us investigate where $MRS_1^{x,y} > MRS_2^{x,y}$.

$$MRS_1^{x,y} > MRS_2^{x,y} \iff 2y_1 > \frac{6 - y_1}{3} \iff y_1 > \frac{6}{7}.$$

Then, if $MRS_1^{x,y} > MRS_2^{x,y} \iff y_1 > \frac{6}{7}$ at an allocation $\boldsymbol{z}$ and either $x_1 = 10$ or $y_1 = 0$, then $\boldsymbol{z}$ is Pareto efficient. (Of course $y_1 = 0$ is not possible given the MRS condition) Thus,

$$(x_1, y_1) = (10, y_1) \text{ where } y_1 > \frac{6}{7} \text{ is Pareto efficient.}$$

Similarly, when $MRS_1^{x,y} < MRS_2^{x,y}$:

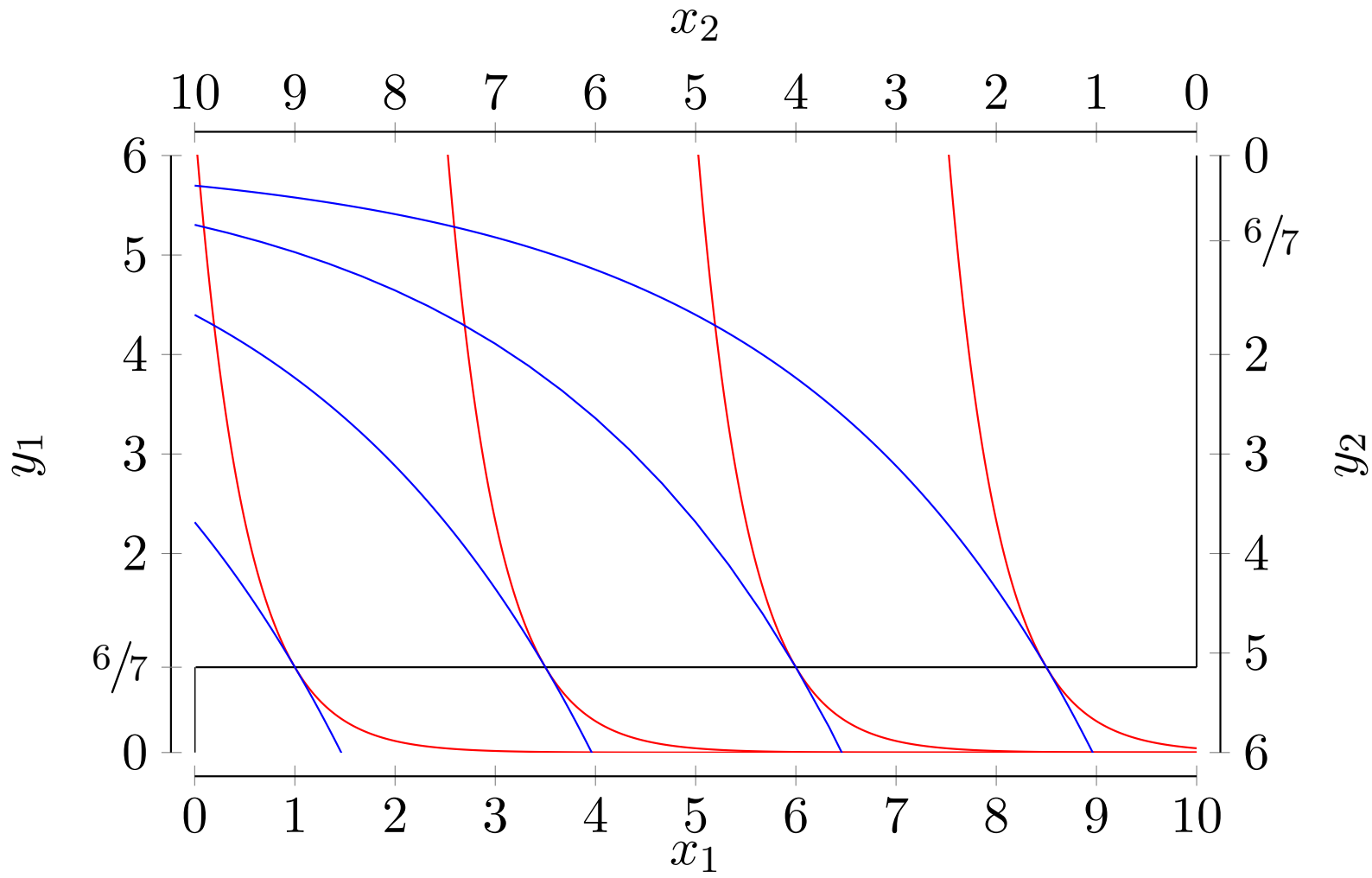
$$MRS_1^{x,y} < MRS_2^{x,y} \iff 2y_1 < \frac{6 - y_1}{3} \iff y_1 < \frac{6}{7}.$$

Then, if $MRS_1^{x,y} < MRS_2^{x,y} \iff y_1 < \frac{6}{7}$ at an allocation $\boldsymbol{z}$ and either $x_1 = 0$ or $y_1 = 6$, then $\boldsymbol{z}$ is Pareto efficient. (Of course $y_1 = 6$ is not possible given the MRS condition) Thus,

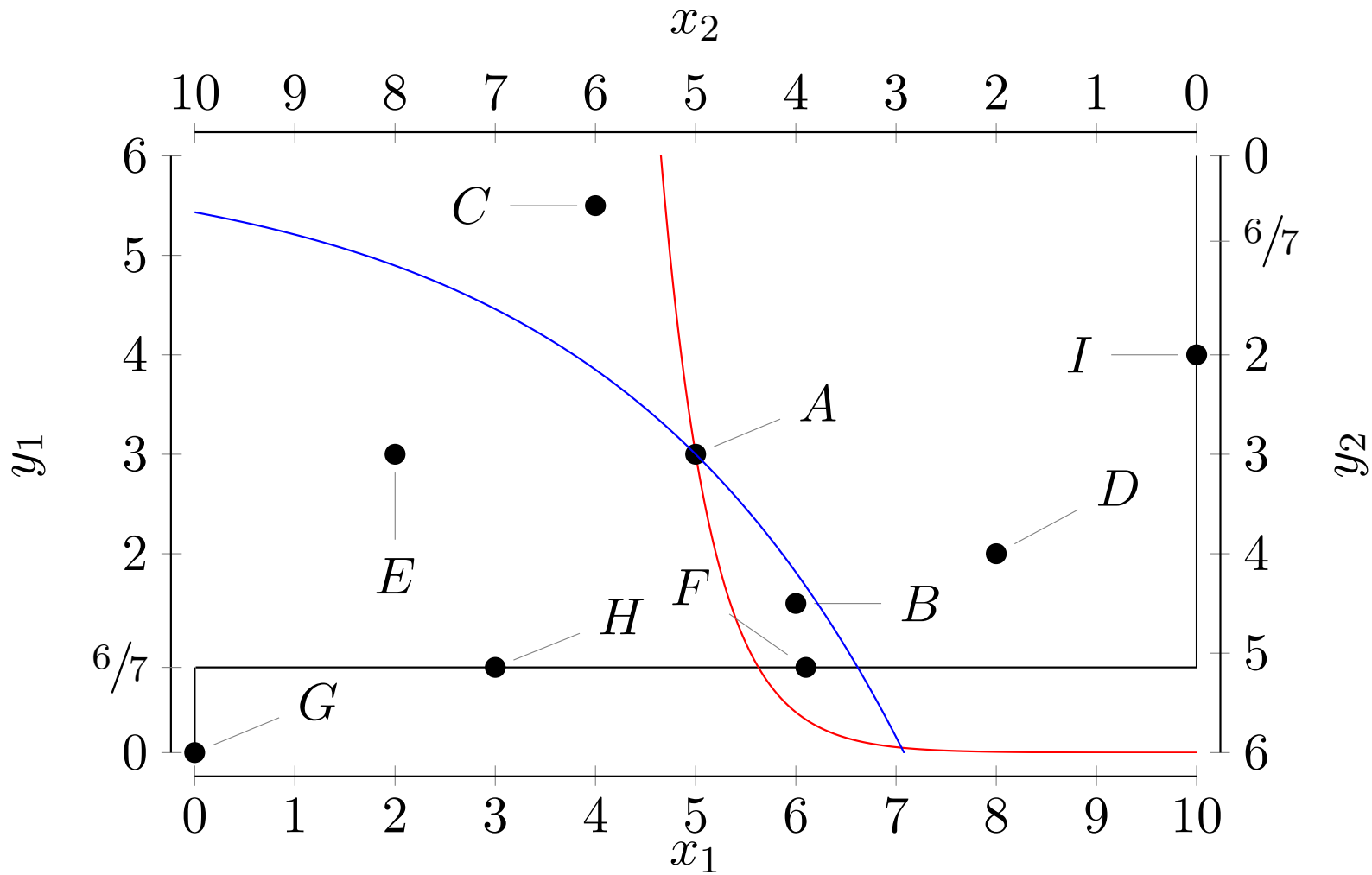
$$(x_1, y_1) = (0, y_1) \text{ where } y_1 < \frac{6}{7} \text{ is Pareto efficient.}$$

To summarize, the Pareto efficient allocations in the given example are:

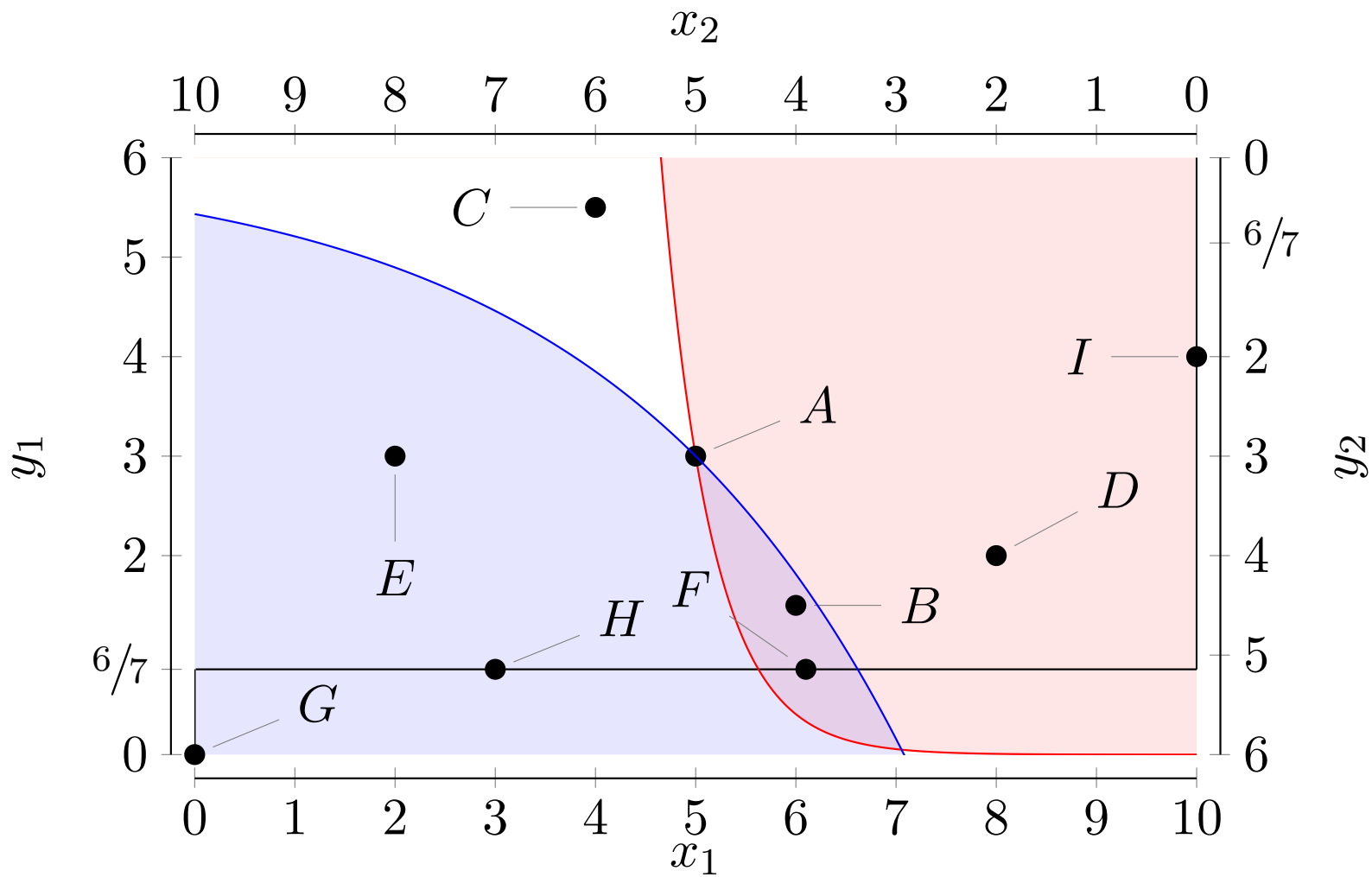
- all allocations $(x_1, y_1) = (x_1, {}^{6}/_{7})$, $(x_2, y_2) = (10 - x_1, {}^{36}/_{7})$;
- all allocations $(x_1, y_1) = (0, y_1)$, $(x_2, y_2) = (10, 6 - y_1)$ where $y_1 < \frac{6}{7}$; and
- all allocations $(x_1, y_1) = (10, y_1)$, $(x_2, y_2) = (0, 6 - y_1)$ where $y_1 > \frac{6}{7}$.


x_2
10 9 8 7 6 5 4 3 2 1 0
y_1
6 5 4 3 2 $6/7$ 0
y_2
0 $6/7$ 2 3 4 5 6
0 1 2 3 4 5 6 7 8 9 10
x_1

Exercise: Which allocations Pareto dominate (or are Pareto superior to) allocation A? Which allocations are Pareto dominated by (or are Pareto inferior to) A? Which allocations are Pareto efficient?



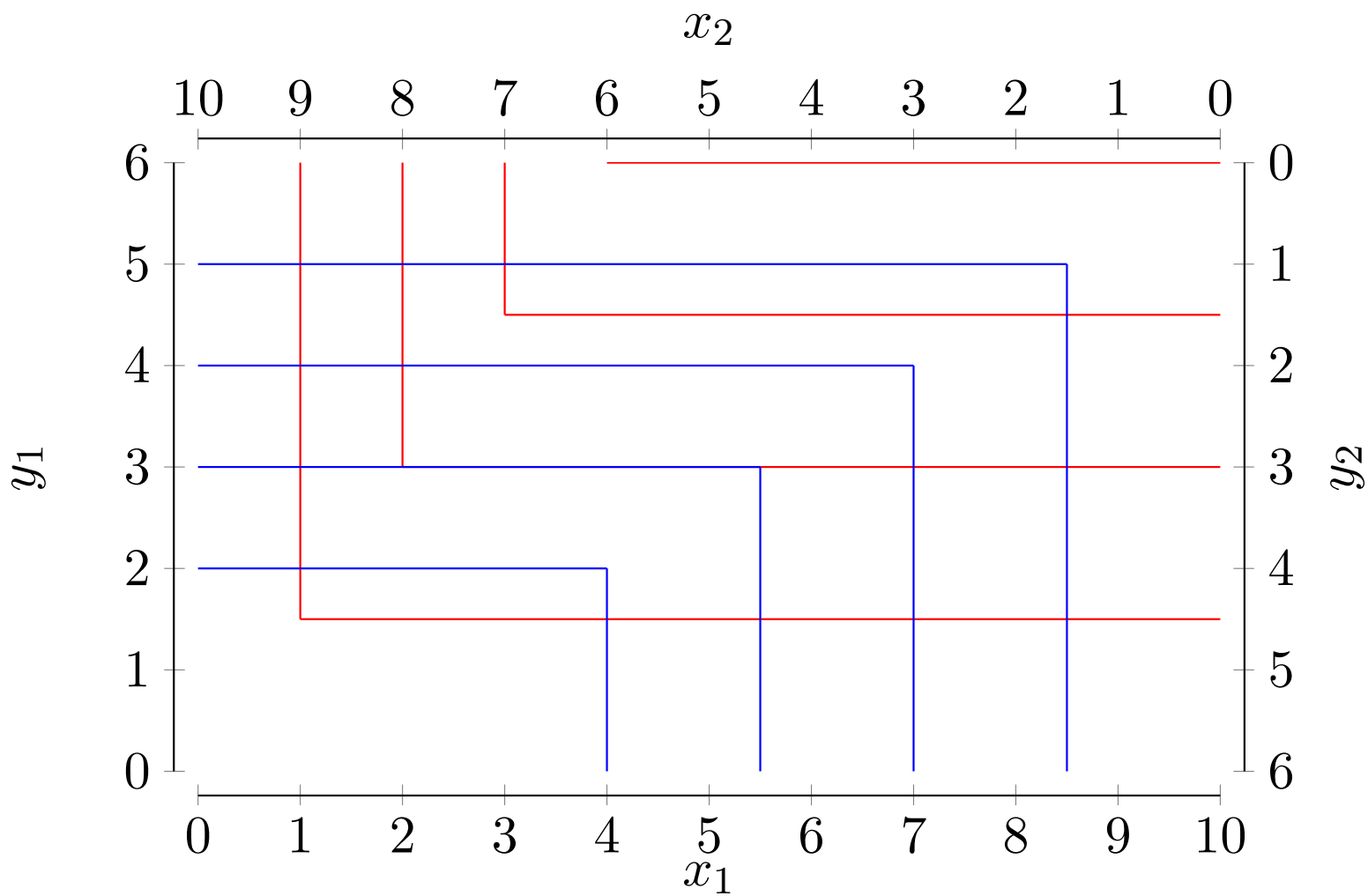
Answer. B and F Pareto dominate A. C is Pareto dominated by A. F, G, H and I are Pareto efficient.



Let's see a different example. Suppose Ann (1) and Bob's (2) preferences are instead given by:

$$u_1(x_1, y_1) = \min\{3x, 2y\},$$
$$u_2(x_2, y_2) = \min\{2x, 3y\}.$$

This is what the indifference curves look like.



Preferences that can be expressed by a utility function of the form

$$u(x, y) = \min\{ax, by\}$$

where a, b are positive constants characterize *Leontieff* preferences. Individuals with these preferences always want x and y in the fixed proportion of 1 unit of x for a/b units of y. If x/y exceeds b/a, then there's an excess of x that goes unappreciated. If x/y falls below b/a, then there's an excess of y that goes unappreciated.

The indifference "curves" for these preferences have their distinct L-shaped format, with the right degree angle positioned at a point where $ax = by$.

Indeed, the indifference "curve" corresponding to a level of utility $\bar{u}$ has its right-degree angle at the point where $ax = by = \bar{u}$, corresponding to the allocation $(\bar{u}/a, \bar{u}/b)$.

Suppose that $ax > by$. Then,

$$MRS^{x,y}(x,y) = \frac{MU^x}{MU^y} = \frac{0}{b} = 0.$$

Suppose that $ax < by$ instead. Then,

$$MRS^{x,y}(x,y) = \frac{MU^x}{MU^y} = \frac{a}{0} = \infty.$$

When $ax = by$, $MRS^{x,y}(x,y) = MRS^{x,y}(x, {}^{ax}/_{b})$ is *undefined*. Think of it as if the indifference curve could have any slope at the kink, ranging from zero to minus infinity.

How do we then find the Pareto efficient points when both Ann and Bob have Leontieff preferences? As before, we will try to find points of "tangency" between the two curves. Tangencies here happen at points where both curves have the same MRS (at zero or infinity), or when one of the two curves is at a kink.[1]

[1] Since the kink has "any slope from zero to minus infinity", informally speaking, it is tangent to anything that has nonpositive slope.

Ann's indifference curves have their kinks along the line characterized by $3x_1 = 2y_1 \iff y_1 = {}^{3x_1}/_2$.

Suppose that $3x_1 > 2y_1 \iff y_1 < {}^{3x_1}/_2$. Then,

$$MRS_1^{x,y} = \frac{MU_1^x}{MU_1^y} = \frac{0}{2} = 0.$$

Suppose that $3x_1 < 2y_1 \iff y_1 > {}^{3x_1}/_2$ instead. Then,

$$MRS_1^{x,y} = \frac{MU_1^x}{MU_1^y} = \frac{3}{0} = \infty.$$

Bob's indifference curves have their kinks along the line characterized by $2x_2 = 3y_2 \implies y_2 = {}^{2x_2}/_3$. Using feasibility, this is equivalent to $6 - y_1 = {}^{2(10 - x_1)}/_3 \iff y_1 = {}^{2x_1}/_3 - {}^{2}/_3$.

Suppose that $2x_2 > 3y_2 \iff y_2 < {}^{2x_2}/_3 \iff y_1 > {}^{2x_1}/_3 - {}^{2}/_3$. Then,

$$MRS_2^{x,y} = \frac{MU_2^x}{MU_2^y} = \frac{0}{3} = 0.$$

Suppose that $2x_2 < 3y_2 \iff y_1 > {}^{2x_1}/_3 - {}^{2}/_3$ instead. Then,

$$MRS_2^{x,y} = \frac{MU_2^x}{MU_2^y} = \frac{2}{0} = \infty.$$

Thus the Pareto efficient points are given by:

- All allocations at either Ann or Bob's indifference curves are at a kink:

$$y_1 = {}^{3x_1}/_2 \text{ and } y_1 = {}^{2x_1}/_3 - {}^{2}/_3$$

- All allocations where $MRS_1^{x,y} = MRS_2^{x,y} = 0$:

$$y_1 < {}^{3x_1}/_2 \text{ and } y_1 > {}^{2x_1}/_3 - {}^{2}/_3$$

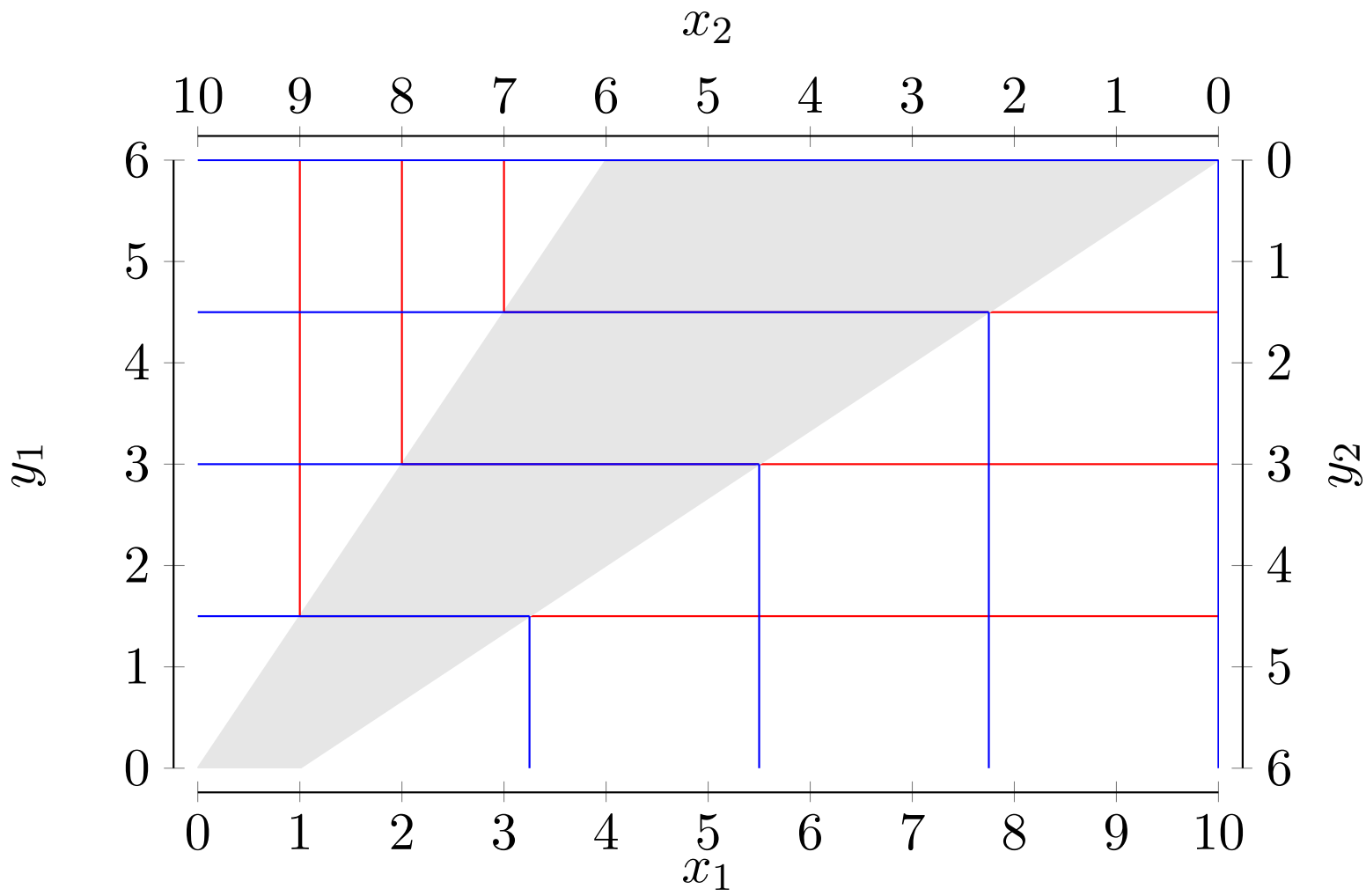
$$\iff {}^{2x_1}/_3 - {}^{2}/_3 < y_1 < {}^{3x_1}/_2.$$

- All allocations where $MRS_1^{x,y} = MRS_2^{x,y} = \infty$:

$$y_1 > {}^{3x_1}/_2 \text{ and } y_1 < {}^{2x_1}/_3 - {}^{2}/_3$$

$$\iff {}^{2x_1}/_3 - {}^{2}/_3 > y_1 > {}^{3x_1}/_2.$$

The last item can be ignored in this example since ${}^{3x_1}/_2 > {}^{2x_1}/_3 - {}^{2}/_3$ for all feasible allocations.


x_2
10 9 8 7 6 5 4 3 2 1 0
y_1
6 5 4 3 2 1 0
y_2
0 1 2 3 4 5 6
0 1 2 3 4 5 6 7 8 9 10
x_1

Let's do another example with Leontieff preferences, to fix ideas.

$$u_1(x_1, y_1) = \min\{2x_1, 7y_1\},$$
$$u_2(x_2, y_2) = \min\{x_2, y_2\}.$$

Ann's indifference curves have their kinks along the line characterized by $2x_1 = 7y_1 \iff y_1 = {}^{2x_1}/_{7}$.

Suppose that $2x_1 > 7y_1 \iff y_1 < {}^{2x_1}/_{7}$. Then,

$$MRS_1^{x,y} = \frac{MU_1^x}{MU_1^y} = \frac{0}{7} = 0.$$

Suppose that $2x_1 < 7y_1 \iff y_1 > {}^{2x_1}/_{7}$ instead. Then,

$$MRS_1^{x,y} = \frac{MU_1^x}{MU_1^y} = \frac{2}{0} = \infty.$$

Bob's indifference curves have their kinks along the line characterized by $x_2 = y_2$. Using feasibility, this is equivalent to $6 - y_1 = 10 - x_1 \iff y_1 = x_1 - 4$.

Suppose that $y_2 < x_2 \iff y_1 > x_1 - 4$. Then,

$$MRS_2^{x,y} = \frac{MU_2^x}{MU_2^y} = \frac{0}{1} = 0.$$

Suppose that $y_2 > x_2 \iff y_1 < x_1 - 4$ instead. Then,

$$MRS_2^{x,y} = \frac{MU_2^x}{MU_2^y} = \frac{1}{0} = \infty.$$

Thus the Pareto efficient points are given by:

- All allocations at either Ann or Bob's indifference curves are at a kink:

$$y_1 = {}^{2x_1}/_7 \text{ and } y_1 = x_1 - 4$$

- All allocations where $MRS_1^{x,y} = MRS_2^{x,y} = 0$:

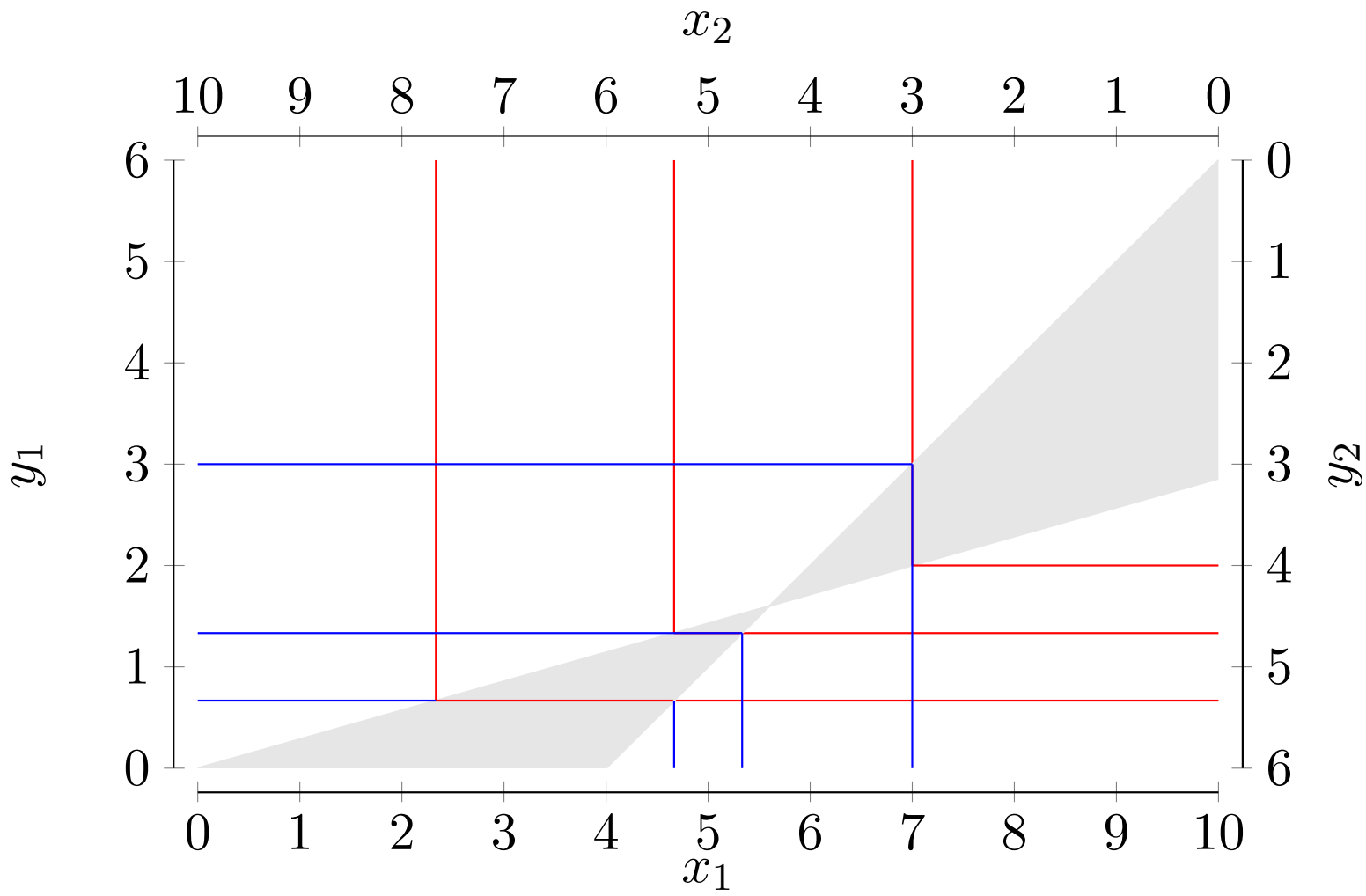
$$y_1 < {}^{2x_1}/_7 \text{ and } y_1 > x_1 - 4$$

$$\iff x_1 - 4 < y_1 < {}^{2x_1}/_7.$$

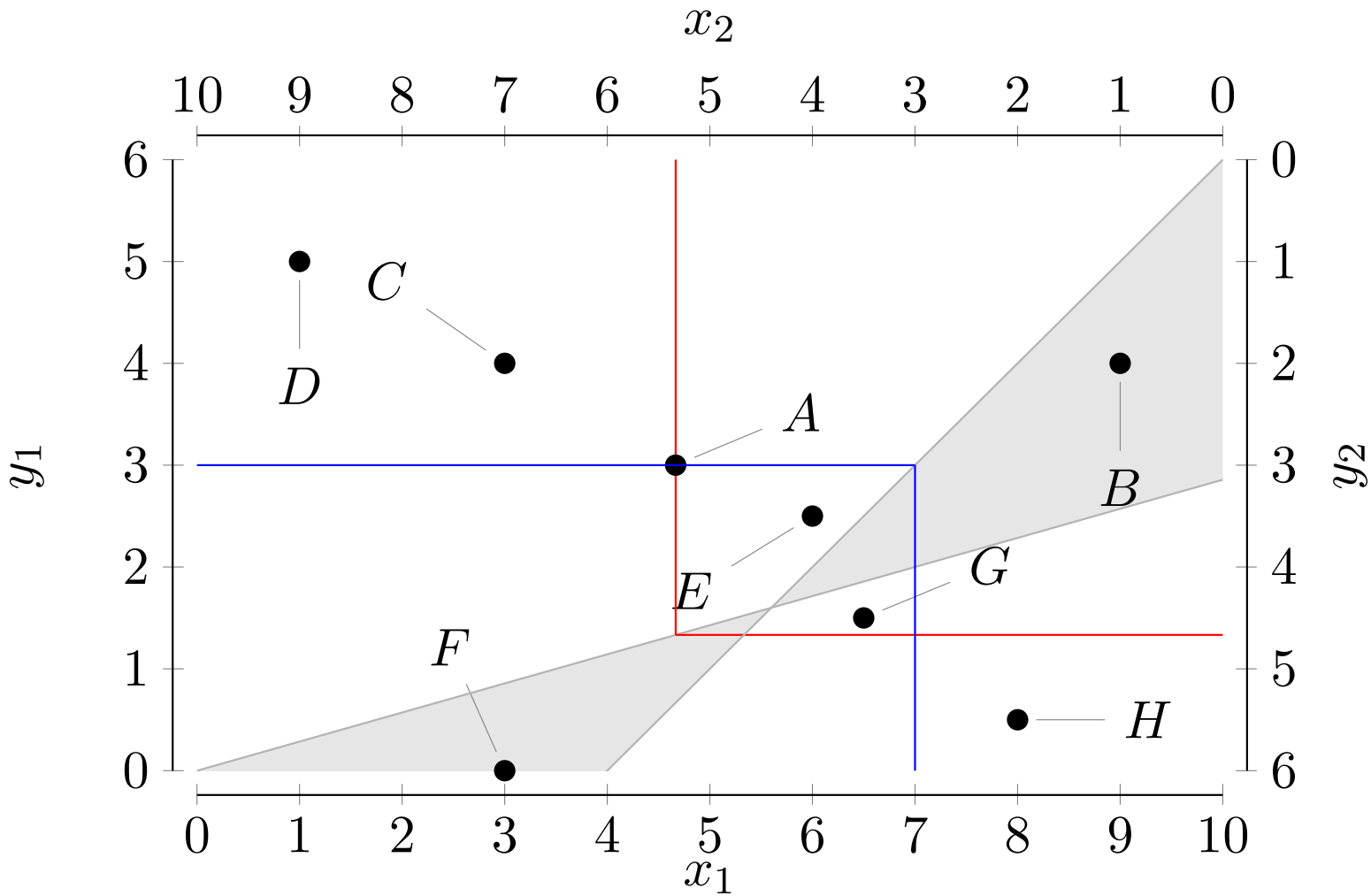
- All allocations where $MRS_1^{x,y} = MRS_2^{x,y} = \infty$:

$$y_1 > {}^{2x_1}/_7 \text{ and } y_1 < x_1 - 4$$

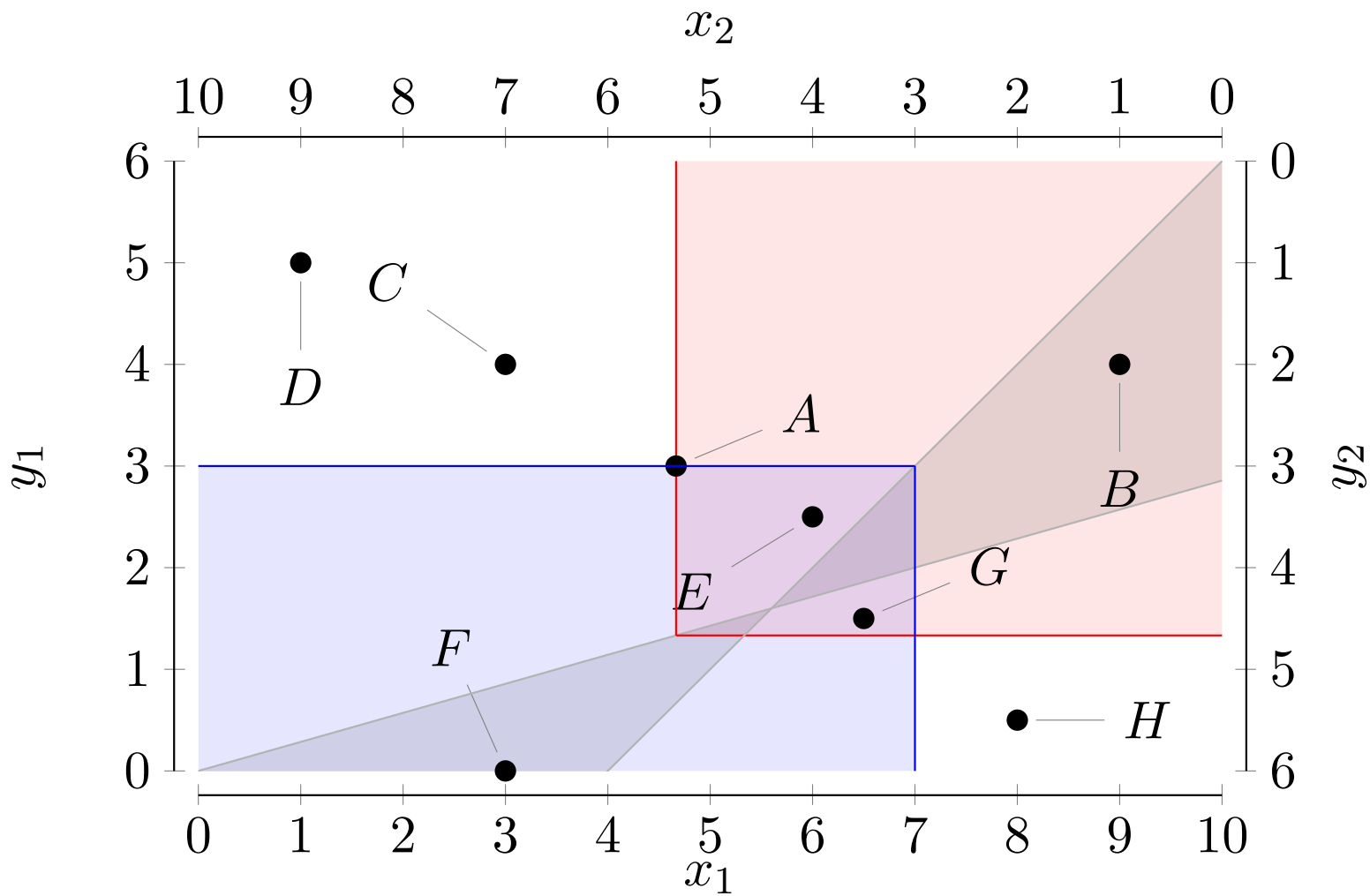
$$\iff x_1 - 4 > y_1 > {}^{2x_1}/_7.$$


x_2
10 9 8 7 6 5 4 3 2 1 0
y_1
6 5 4 3 2 1 0
y_2
0 1 2 3 4 5 6
0 1 2 3 4 5 6 7 8 9 10
x_1

Exercise: Which allocations Pareto dominate (or are Pareto superior to) allocation A? Which allocations are Pareto dominated by (or are Pareto inferior to) A? Which allocations are Pareto efficient?



Answer: E and G Pareto dominate A. C, D and H are Pareto dominated by A. B and F are Pareto efficient.

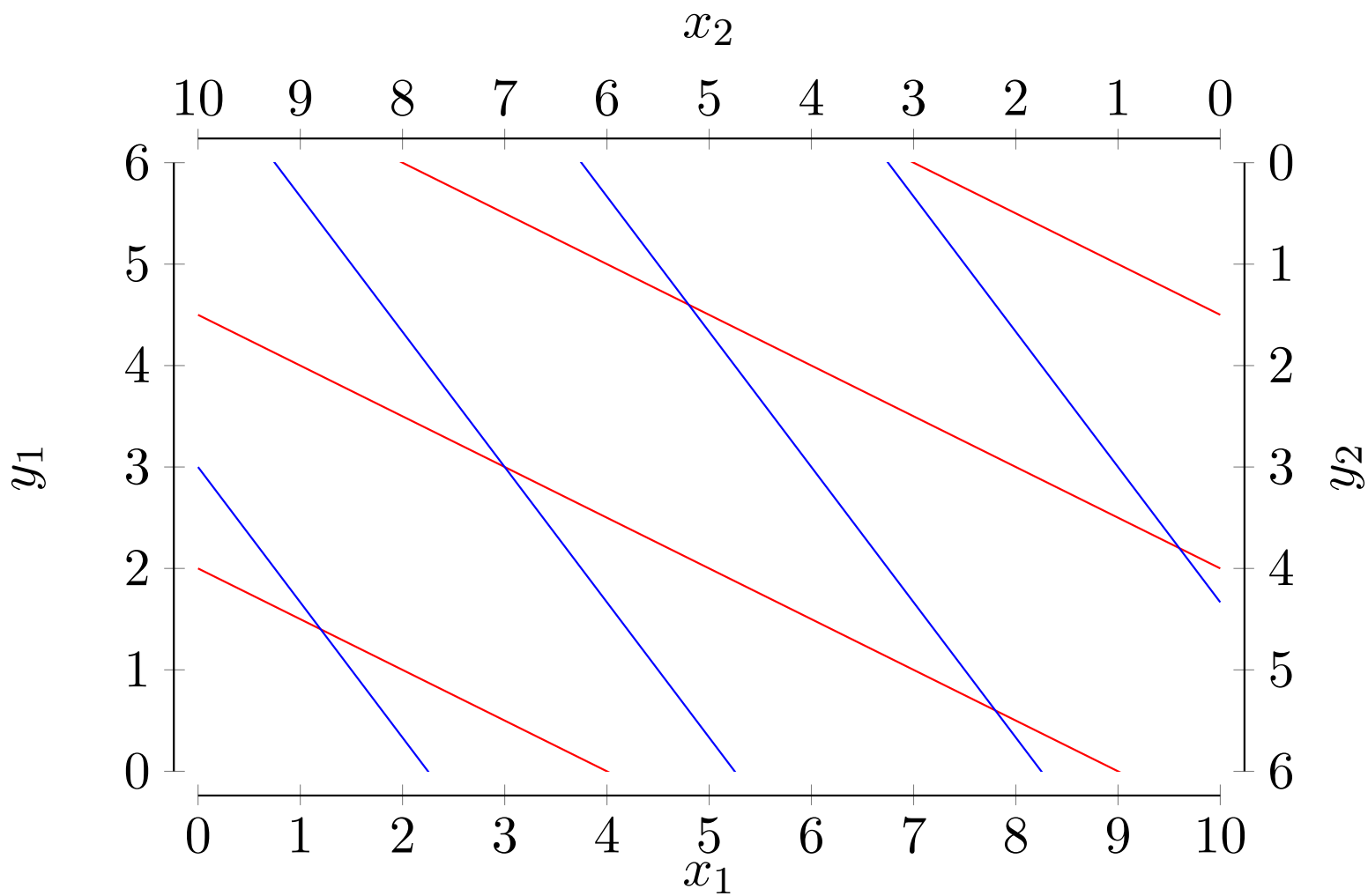


Perfect substitutes

Suppose now that Ann (1) and Bob's (2) preferences are given by the utility functions below.

$$u_1(x_1, y_1) = x + 2y$$
$$u_2(x_2, y_2) = 4x + 3y.$$

This is what the indifference curves look like.



A utility function of the form

$$u(x, y) = ax + by,$$

where a, b are positive constants, characterize preferences for *perfect substitutes*. Individuals with these preferences value one extra unit of x as much as a/b extra units of y, regardless of how much x and y they already have.

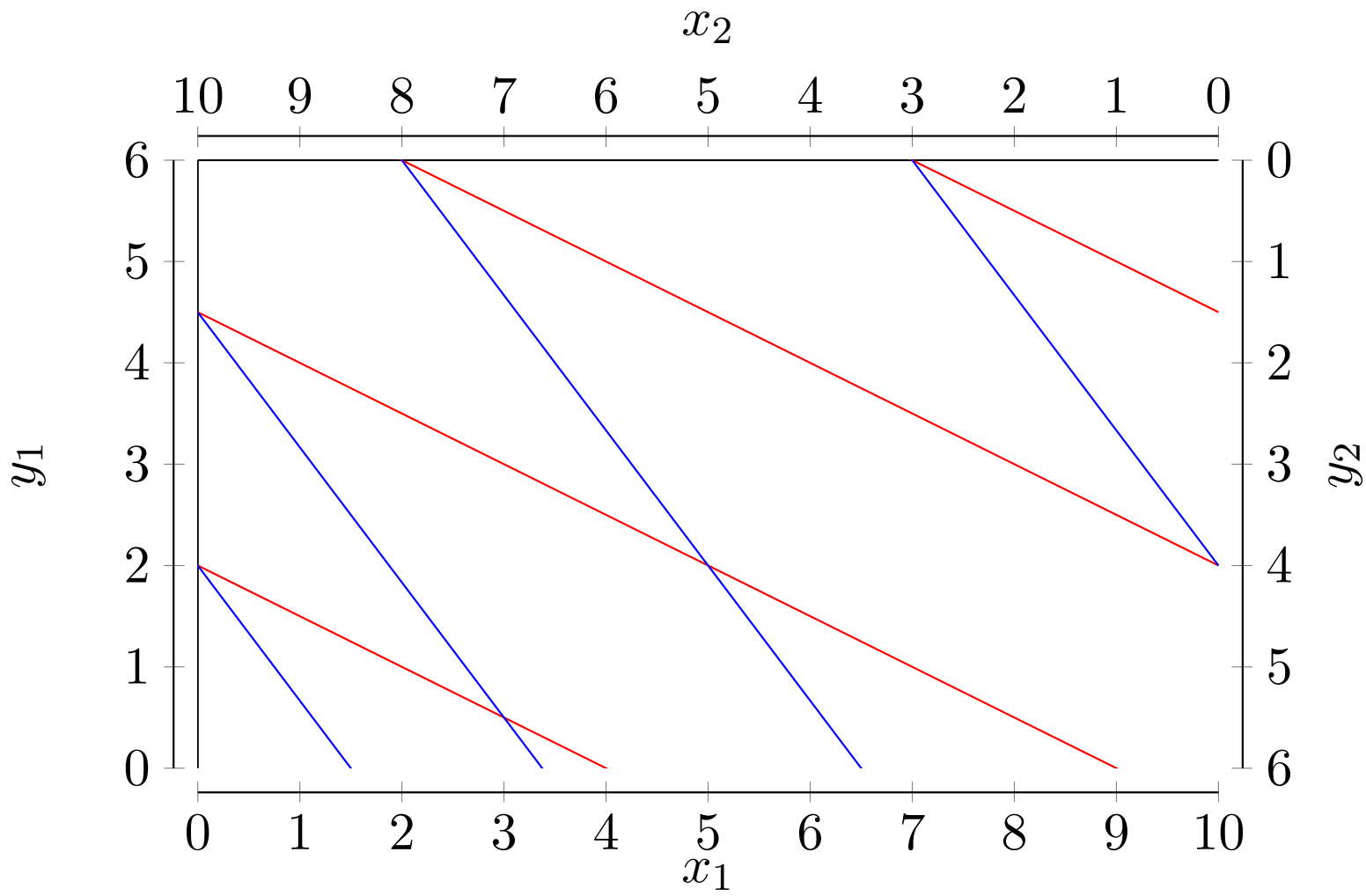
The main defining characteristic of these preferences is the fact that its marginal rate of substitution is constant in x and y:

$$MRS^{x,y} = \frac{MU^x}{MU^y} = \frac{a}{b}.$$

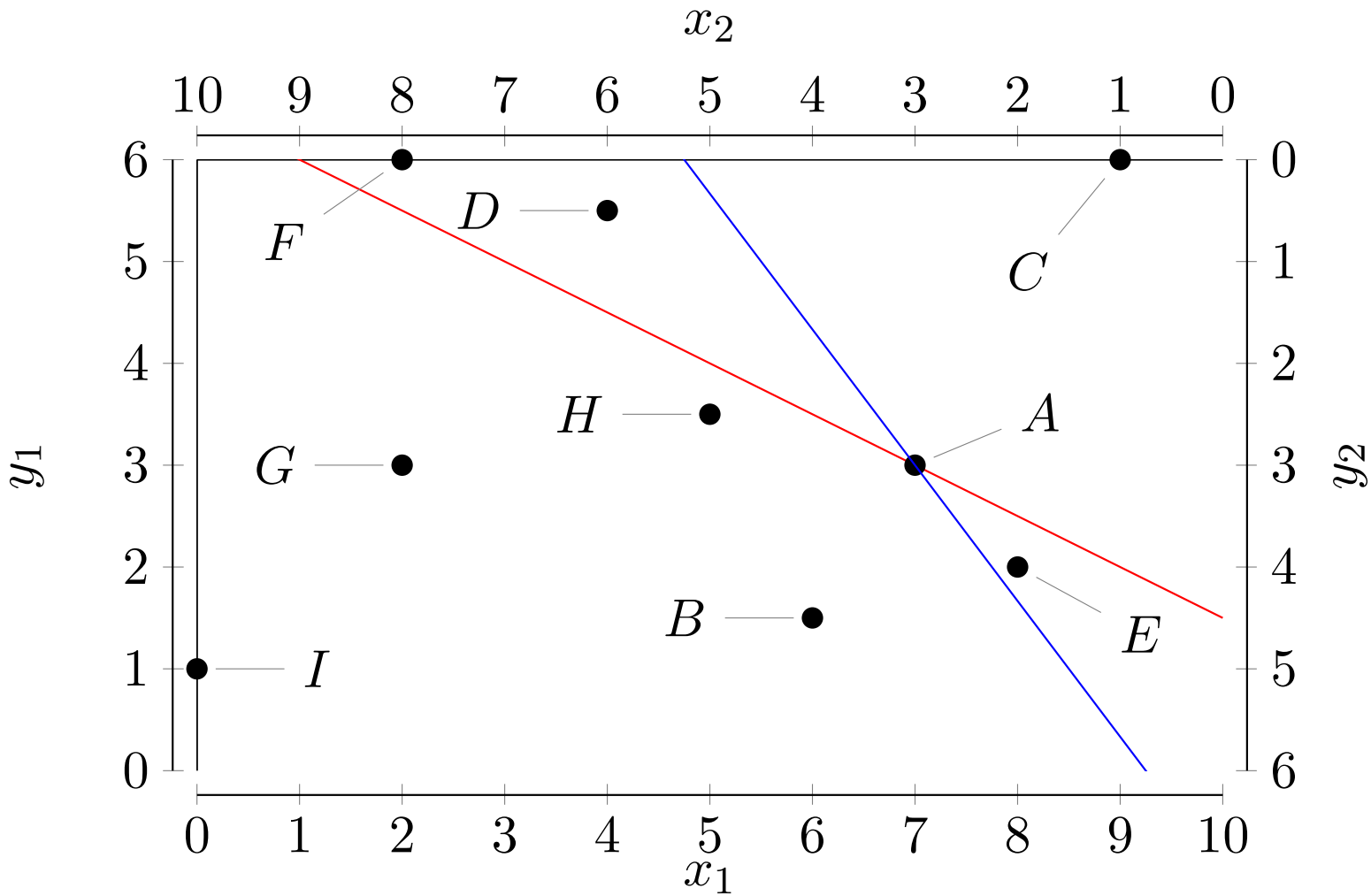
In the example we are working with,

$$MRS_1^{x,y} = \frac{1}{2} \quad \text{and} \quad MRS_2^{x,y} = \frac{4}{3}$$

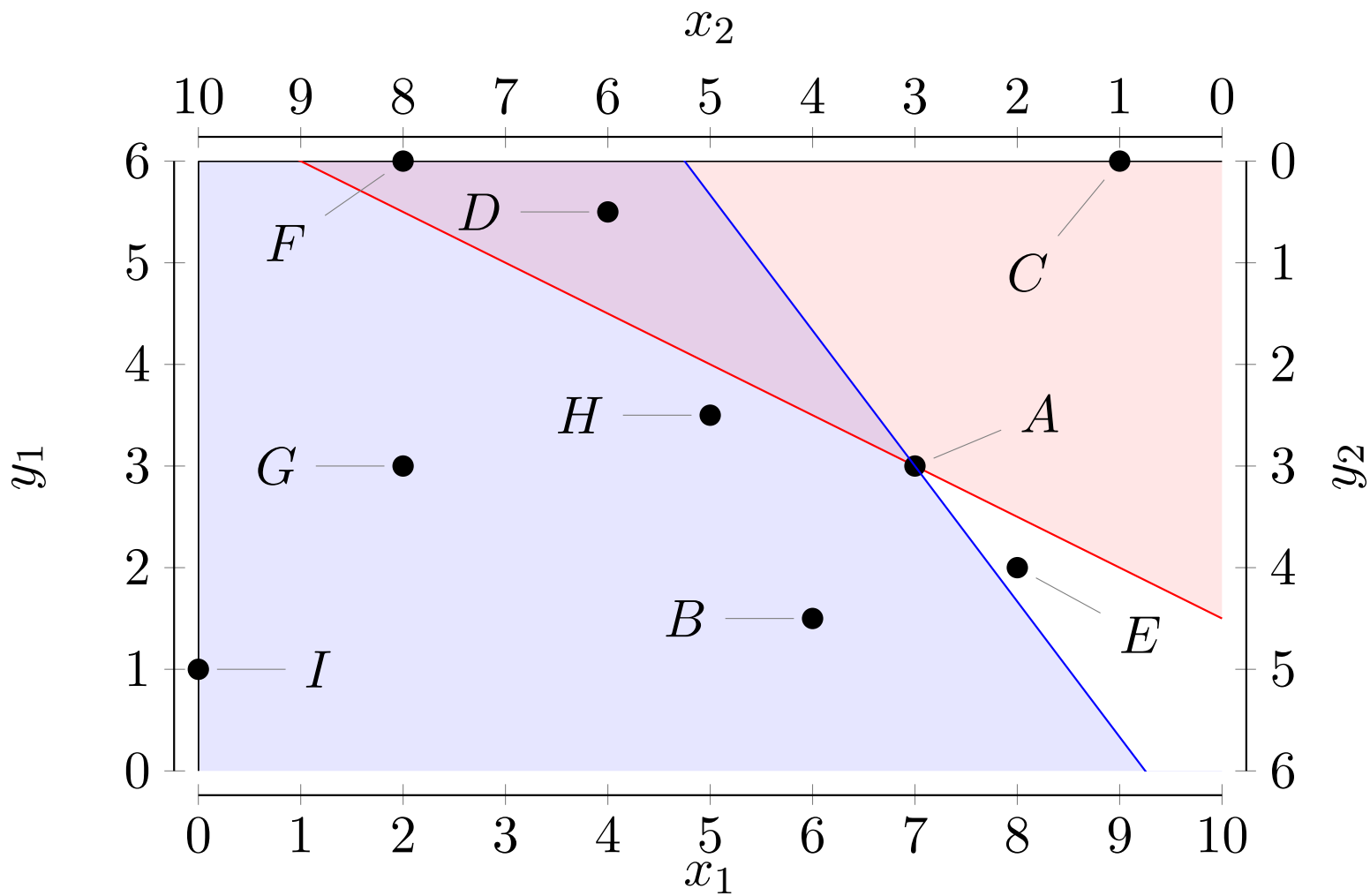
That is, $MRS_1^{x,y} < MRS_2^{x,y}$ for all values of x, y: Ann and Bob would be willing to engage in trades where Ann is always willing to give up on some x in exchange for some extra y, and Bob would be willing to engage in such a trade. Thus, Pareto efficiency is only possible if $x_1 = 0$ or $y_1 = e^y = 6$.


x_2
10 9 8 7 6 5 4 3 2 1 0
6 5 4 3 2 1 0
y_1
y_2
0 1 2 3 4 5 6
0 1 2 3 4 5 6 7 8 9 10
x_1

Exercise: Which allocations Pareto dominate (or are Pareto superior to) allocation A? Which allocations are Pareto dominated by (or are Pareto inferior to) A? Which allocations are Pareto efficient?



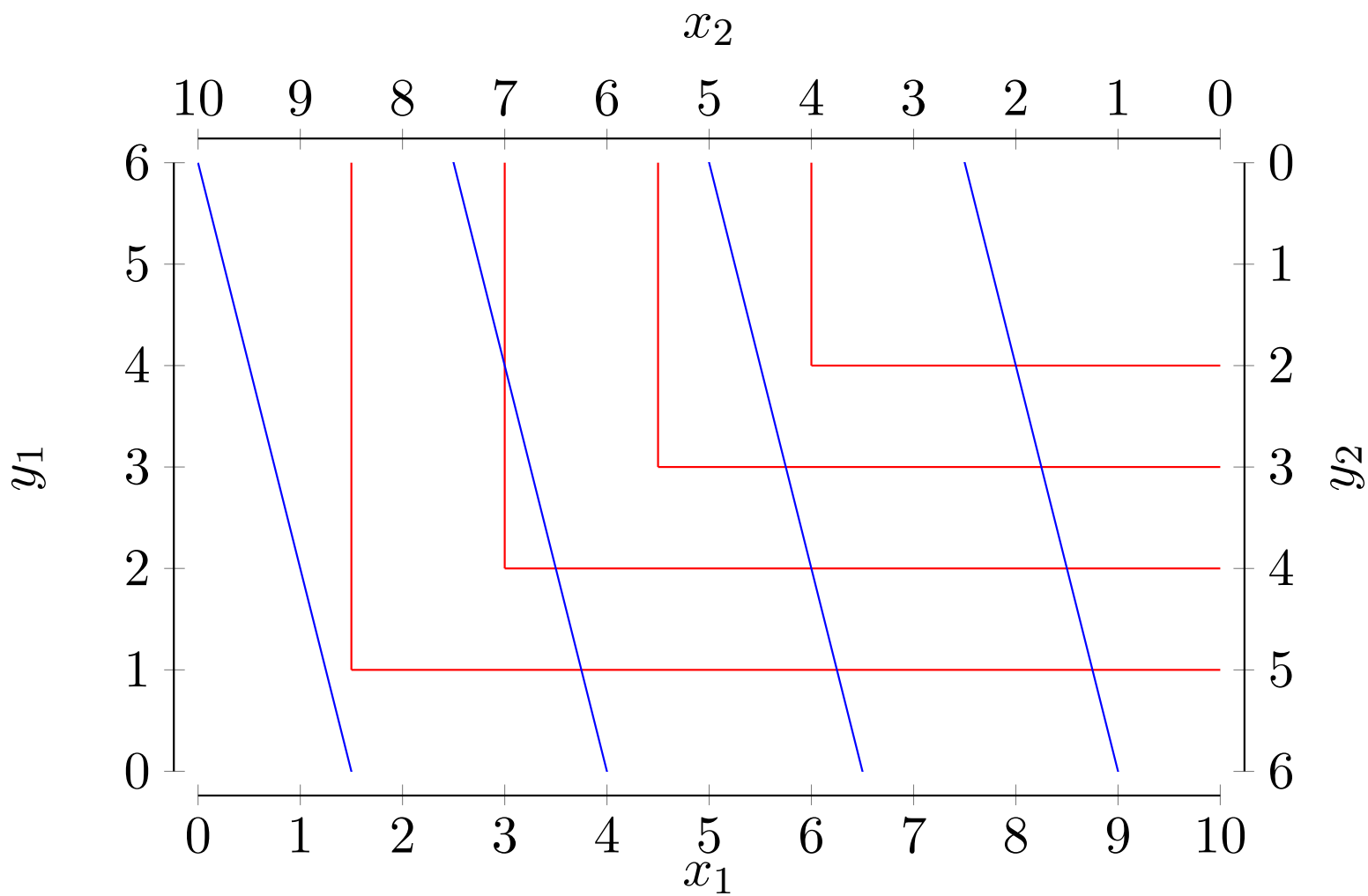
Answer: D and F Pareto dominate A. A Pareto dominates E. C, F and I are Pareto efficient.



Suppose now that Ann (1) and Bob's (2) preferences are given by the utility functions below.

$$u_1(x_1, y_1) = \min\{2x, 3y\}$$
$$u_2(x_2, y_2) = 4x + y.$$

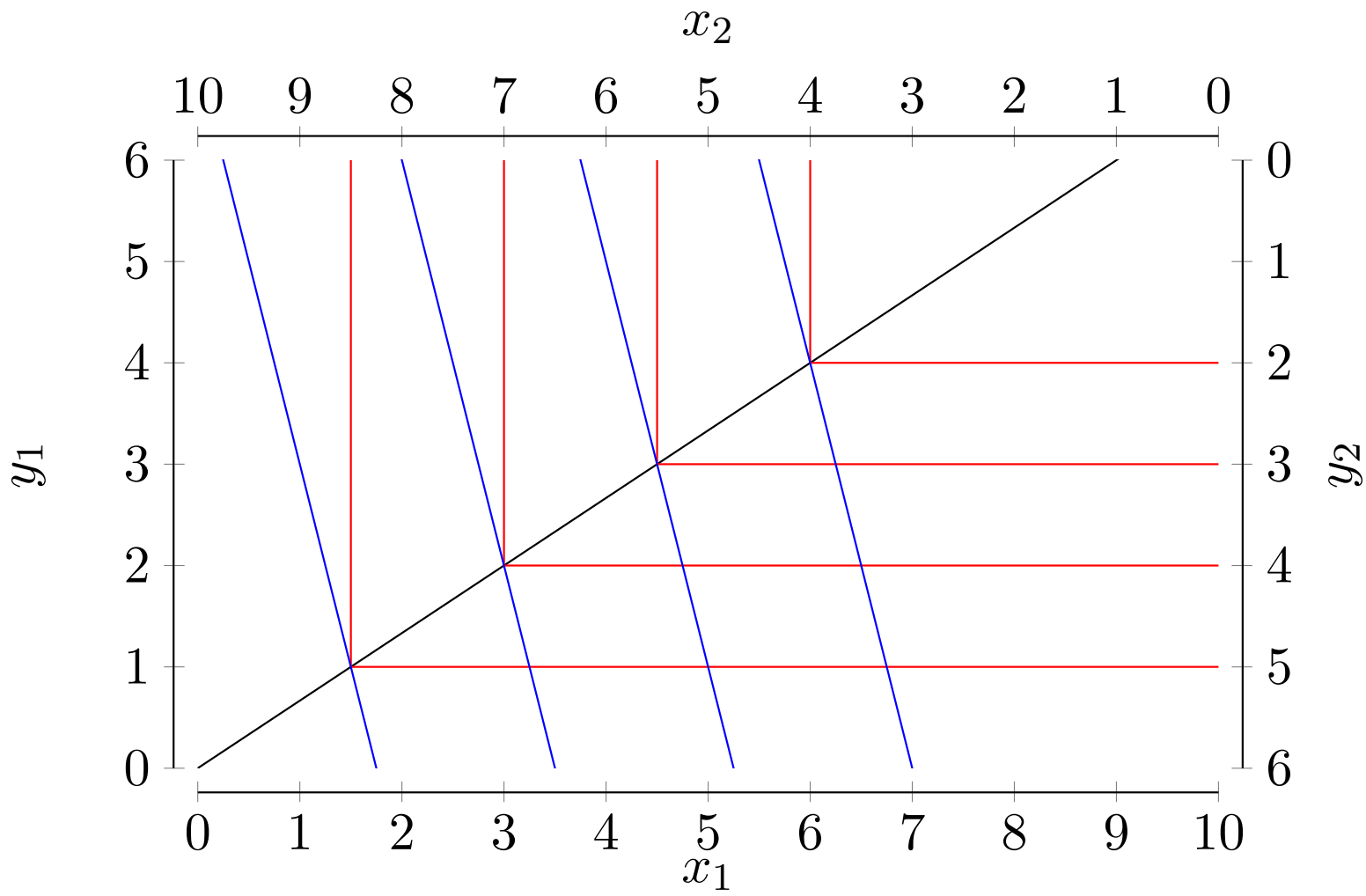
This is what the indifference curves look like.



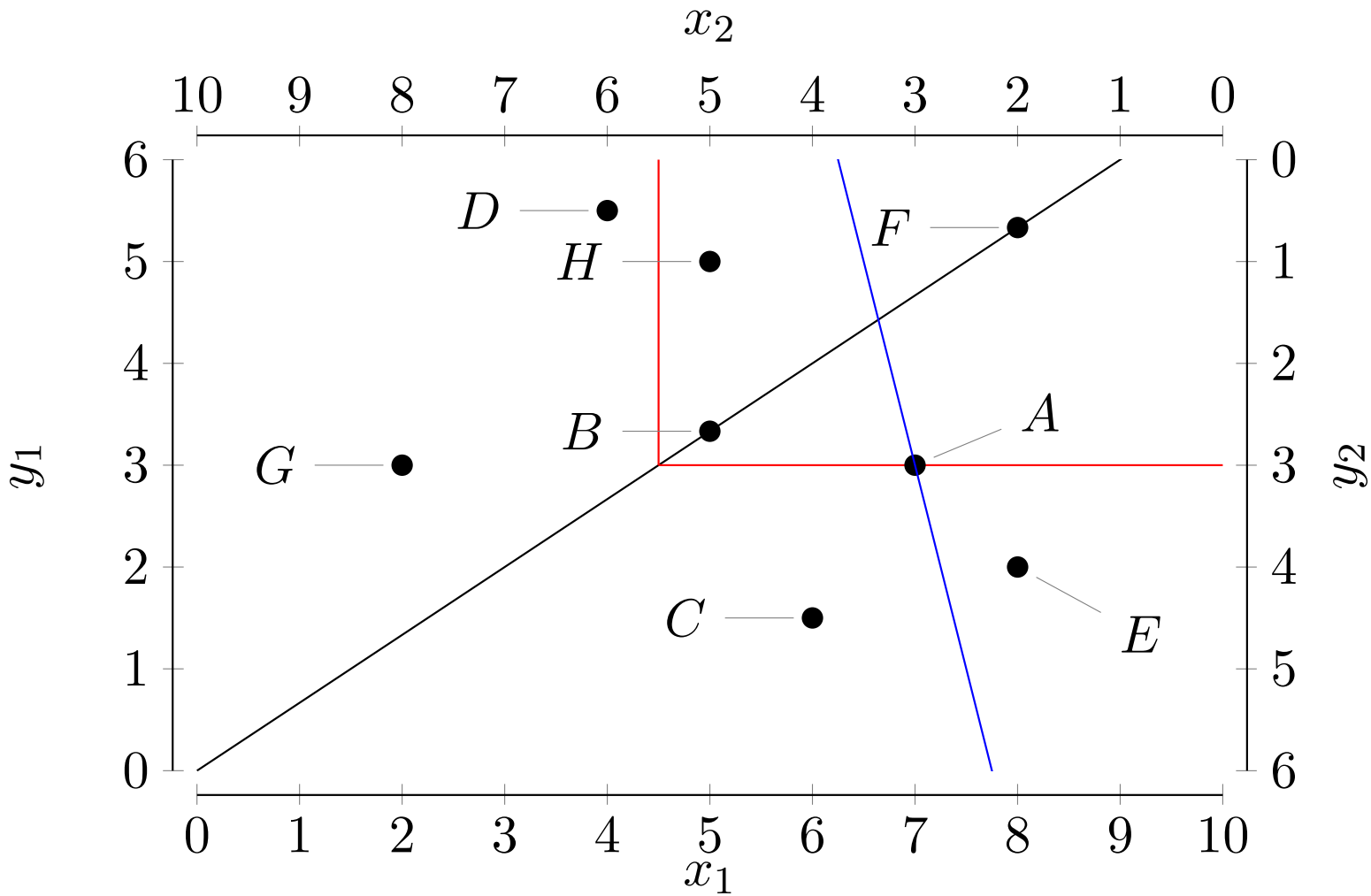
We first recognize the interior Pareto efficient points. These must lie at the kinks of Ann's indifference curves. Otherwise, there would be some extra x (or extra y) that could be transferred from Ann to Bob which would keep Ann just as happy, and make Bob strictly better off. Thus, we are looking at the points where $2x_1 = 3y_1$, or $y_1 = {}^{2x_1}/_{3}$.

There aren't any Pareto efficient points at the boundaries this time.[2]

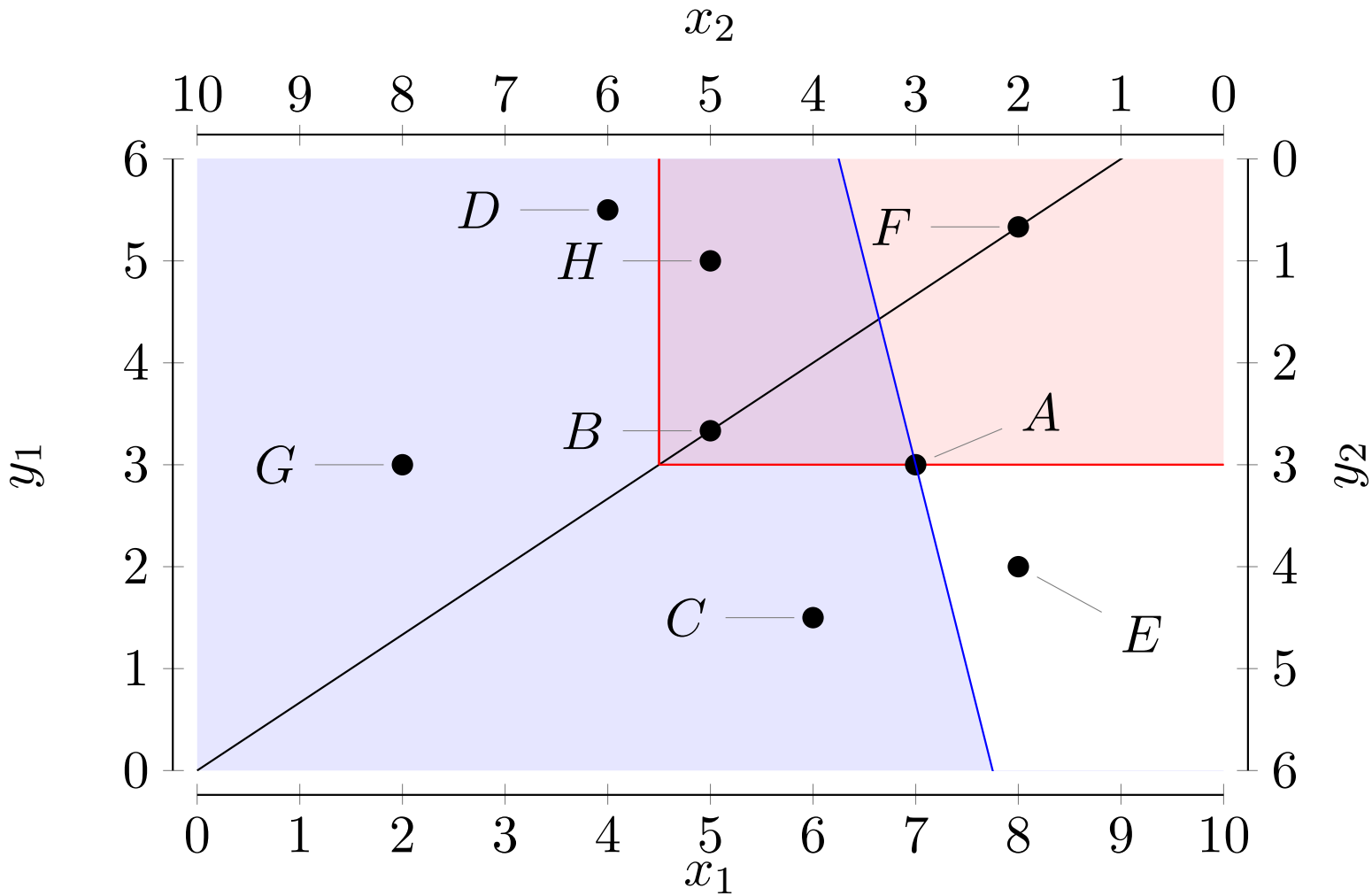
[2] You might think that the points between $(x_1, y_1) = (9, 6)$ and $(10, 6)$ are suspicious... As it turns out, here none of these points is efficient. To see why, note that Ann would be just as happy if we gave her $(9, 6)$ or $(9.5, 6)$ (or $(9.8, 6)$, $(9.9, 6)$, $(10, 6)$...). So $(9, 6)$ actually **Pareto dominates** $(9 + \varepsilon, 6)$ (for $0 \leq \varepsilon \leq 1$), as it is *strictly* better for Bob and just as good for Ann.


x_2
10 9 8 7 6 5 4 3 2 1 0
y_1
6 5 4 3 2 1 0
y_2
0 1 2 3 4 5 6
0 1 2 3 4 5 6 7 8 9 10
x_1

Exercise: Which allocations Pareto dominate (or are Pareto superior to) allocation A? Which allocations are Pareto dominated by (or are Pareto inferior to) A? Which allocations are Pareto efficient?



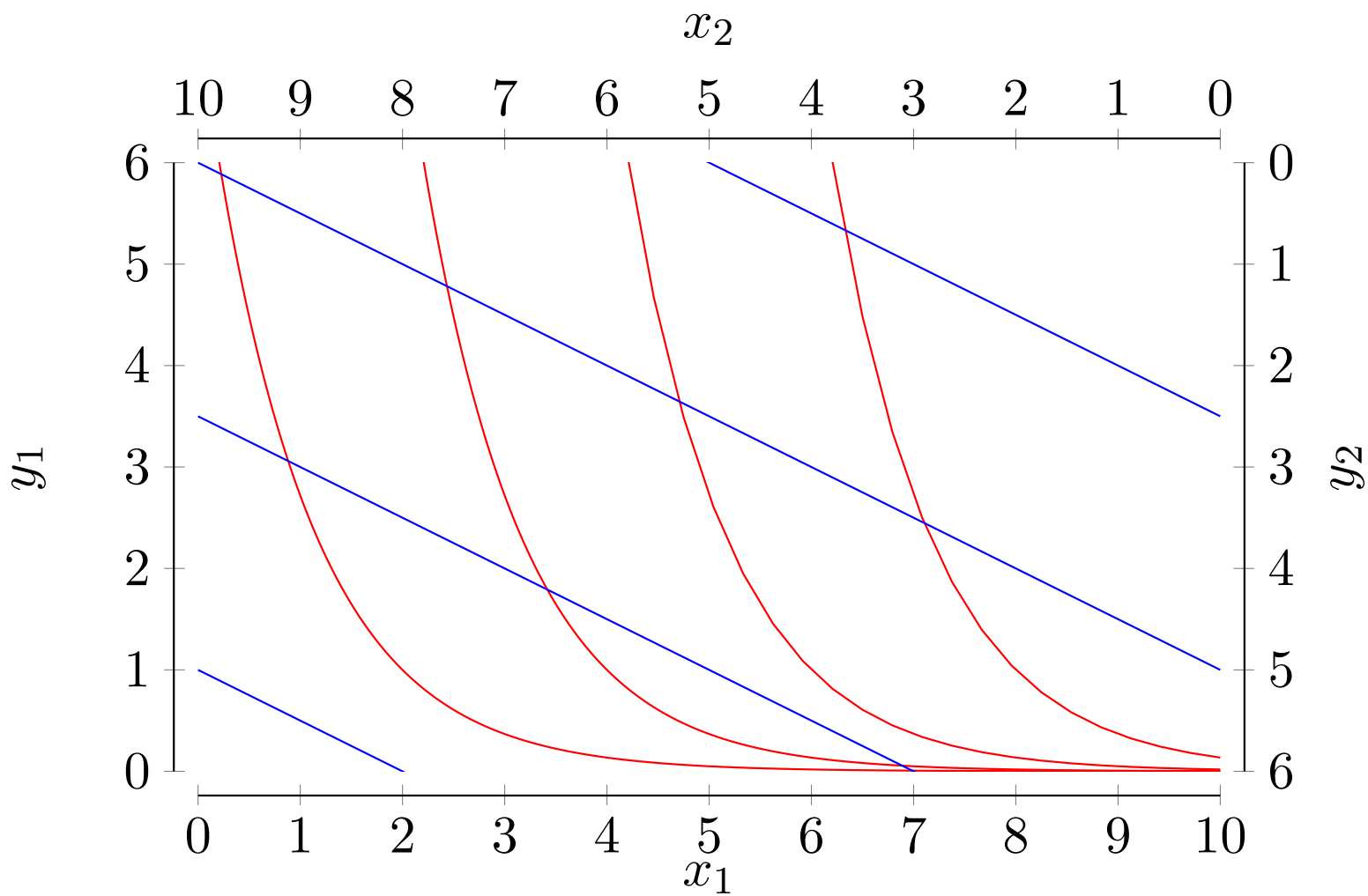
Answer: A Pareto dominates E. B and H Pareto dominate A. B and F are Pareto efficient.



Let us look at one final example.

$$u_1(x_1, y_1) = x + \ln(y)$$
$$u_2(x_2, y_2) = x + 2y.$$

This is what the indifference curves look like.

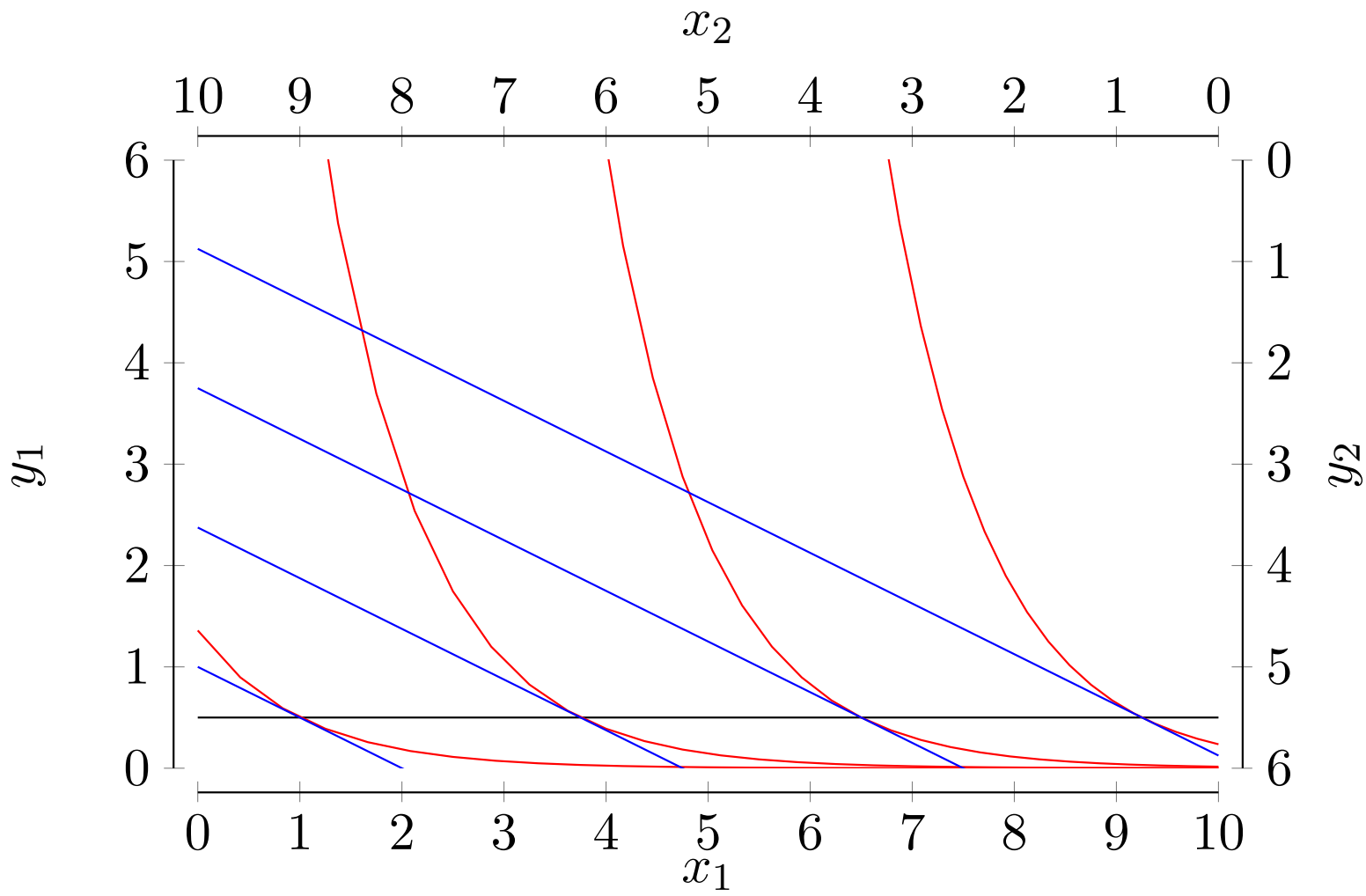


Recall that perfect substitute types of preferences are characterized by constant marginal rates of substitution. Quasilinear preferences have marginal rates of substitution that depend only on the quasilinear good y.

$$MRS_1^{x,y} = \frac{MU_1^x}{MU_1^y} = \frac{1}{1/y_1} = y_1,$$
$$MRS_2^{x,y} = \frac{MU_2^x}{MU_2^y} = \frac{1}{2}.$$

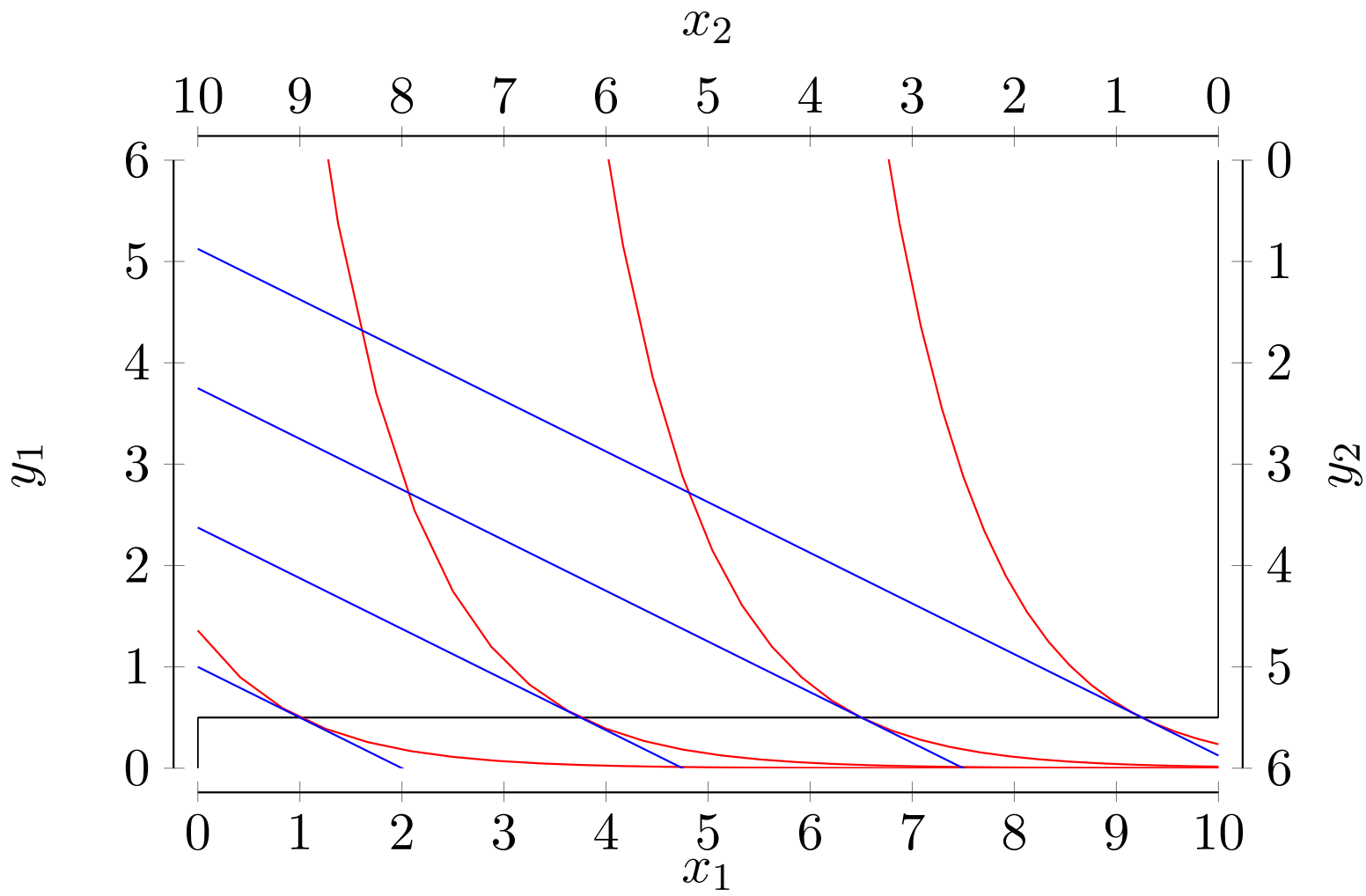
The two marginal rates of substitution are equal whenever $y_1 = 1/2$.

This gives us the interior Pareto efficient points:


x_2
10 9 8 7 6 5 4 3 2 1 0
y_1
0 1 2 3 4 5 6
y_2
0 1 2 3 4 5 6
0 1 2 3 4 5 6 7 8 9 10
x_1

Now, for the boundaries, note that whenever $y_1 > 1/2$, then Ann value for x compared to y is higher than Bob's. That means Ann would be willing to acquire x from Bob in exchange for some of her y, and Bob would be happy to accept such a trade. Thus $y_1 > 1/2$ can only be Pareto efficient if such trades cannot happen, i.e. when $x_1 = 10$.

Similarly, $y_1 < 1/2$ is Pareto efficient only if $x_1 = 0$.


x_2
10 9 8 7 6 5 4 3 2 1 0
y_1
0 1 2 3 4 5 6
y_2
0 1 2 3 4 5 6
0 1 2 3 4 5 6 7 8 9 10
x_1