Notes on All-Pay Auctions

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1 Introduction

In a (first-price) all-pay auction, each bidder $i \in I$ submits a non-negative sealed bid x_i for an item valued by player i at v_i. It is similar to a standard (winner pay) first-price auction, except that losers must also pay the auctioneer their bids.

The all pay auction is widely used in economics because it captures the essential elements of contests. It has been used to model lobbying, technological competition and R&D races, political campaigns, tournaments and job promotion contests.

In these notes we will show how to construct the mixed-strategy Nash equilibrium of a two-player complete-information all-pay auction with common values.[1] The main reference for this material is Baye et al. [1996] (and the working paper version in Baye et al. [1990]).

2 Two players, full information and common values

Assume there are only two players, i.e. $I = \{1, 2\}$, and symmetric common valuations normalized to 1, i.e. $v_1 = v_2 = 1$. Each player's strategy consists of a bid $x \in [0, B]$, where $B > 1$. Under complete information, the payoff to

[1] The argument presented here can be generalized; see Baye et al. [1996].

player i with opponent $j \neq i$ is given by:

$$u_i(x_1, x_2) = \begin{cases} -x_i & \text{if } x_j > x_i, \\ \frac{1}{2} - x_i & \text{if } x_i = x_j, \\ 1 - x_i & \text{if } x_i > x_j. \end{cases} \tag{1}$$

Formally, this game is defined by:

$$\Gamma := \Big\{ I, ([0,B])_{i \in I}, (u_i)_{i \in I} \Big\},$$

and we want to compute the unique Nash equilibrium of this game.

Theorem 2.1. *The first-price, sealed bid, two-player all-pay auction with complete information and common value $v = 1$ has a unique Nash equilibrium in which both players choose the same mixed strategy with cummulative distribution G:*

$$\forall x \in \operatorname{supp} G = [0,1], \quad G(x) = x. \tag{2}$$

The proof goes a lot more smoothly if we prove a bunch of lemmas first. I'll do that in *excessive* detail, on purpose. Here's a roadmap of the steps:

1. We can't have equilibria in pure strategies (Lemma 2.1.1).
2. Both players' mixed strategies must lie between 0 and 1 (Lemma 2.1.2)
3. The bottom of their supports has to be at 0. No player has a mass-point at zero (Lemma 2.1.3 and Lemma 2.1.4).
4. Both players end up with zero expected utility in equilibrium (Lemma 2.1.5).
5. The top of their supports is at 1 (Lemma 2.1.6).
6. No player has any mass-points in $(0, 1]$ (Lemma 2.1.7).
7. Both players randomize continuously on $[0, 1]$ support, i.e. the distributions have no gaps (Lemma 2.1.8).

Lemma 2.1.1. *There does not exist a Nash equilibrium in pure strategies.*

Proof. Fix $\boldsymbol{x}^* = (x_1^*, x_2^*)$, where $x_i \in [0, B]$ for each $i \in I$. We show that $\boldsymbol{x}^*$ can't be a Nash equilibrium of the all-pay auction. The goal is to find that, no matter what x_1^*, x_2^* are, someone is not playing a best response, i.e. there's a superior alternative.[2] This will be a bit tedious, as we have to go case by case.

1. If $x_i^* > v_i = 1$, then $x_i^* \notin \mathrm{BR}_i(x_j^*)$. The reason is simple: i's payoff with that bid is at most $1 - x_1^* < 0$. Clearly, she's better off by choosing $x_i = 0$ and guaranteeing a payoff of zero.

2. If $0 \le x_i^* < x_j^* \le 1$, j would prefer choosing something closer to x_i^*: why spend more money than necessary to win? In symbols, for all $\varepsilon < x_j^* - x_i^*$,

$$u_j(\boldsymbol{x}^*) = 1 - x_j^* < 1 - (x_j^* - \varepsilon) = u(x_i^*, x_j^* - \varepsilon),$$

   so $x_j^* \notin \mathrm{BR}_j(x_i^*)$.

3. If $0 \le x_i^* = x_j^* < 1$, then i (or j) would prefer to pick $x_i^* + \varepsilon < 1$, for $\varepsilon < 1/2$. If they are splitting the prize, why should i not increase her bid (very) slightly and guarantee a win? In symbols, for all $\varepsilon < 1/2$,

$$u_i(\boldsymbol{x}^*) = 1/2 - x_i^* < 1 - (x_i^* + \varepsilon) = u(x_i^* + \varepsilon, x_j^*),$$

   so $x_i^* \notin \mathrm{BR}_i(x_j^*)$.

4. Finally, if $x_i^* = x_j^* = 1$, then i (or j) prefers to choose 0 rather than splitting the prize at 1. Right now each player is spending 1 to receive half a prize, valued at $1/2$, netting $-1/2$ payoff. Zero seems like a better choice. In symbols,

$$u_i(0, 1) = 0 > 1/2 - 1 = u(1, 1),$$

   so $1 \notin \mathrm{BR}_j(1)$.

■

Now that we know we will be dealing with mixed strategies, we first compute their supports. Let $\overline{x}_i$ and $\underline{x}_i$ denote the top and bottom of i's equilibrium strategy support, respectively.

[2] We don't need to find the "best" alternative in each case, just a better one is enough.

Lemma 2.1.2. *For all* $i \in I$, $1 \geq \overline{x}_i \geq \underline{x}_i \geq 0$.

Proof. Bids have to be non-negative by the definition of the two players' strategy spaces. Moreover, by setting $x_i = 0$, each player can guarantee at least zero, ruling out bids greater than 1. ■

Lemma 2.1.3. *If* $\underline{x}_i \geq \underline{x}_j$*, then* $\underline{x}_j = 0$*. Moreover, j will place no density on the interval* $(0, \underline{x}_i)$*. If, in addition, i has no atom at* $\underline{x}_j$*, then j will place no density on the interval* $(0, \underline{x}_i]$.[3]

Proof. Let $u_j(x_j, G_i)$ denote j's payoff to bidding x_j when i chooses the cumulative distribution function G_i. Then, if i puts zero mass on $\underline{x}_j$,

$$u_j(\underline{x}_j, G_i) = -\underline{x}_j,$$

since playing $\underline{x}_j$ ensures j loses for sure. The same is true for any $x_j < \underline{x}_i$:

$$\forall x_j \in [\underline{x}_j, \underline{x}_i), \quad u_j(x_j, G_i) = -x_j.$$

The above means that $\underline{x}_j$ must be zero, and that j will never want to play anything in $(0, \underline{x}_i)$ – i.e., j's chosen density should be zero in that interval.

If in addition i puts no atom at $\underline{x}_i$, then we can strengthen the above statement:

$$\forall x_j \in [\underline{x}_j, \underline{x}_i], \quad u_j(x_j, G_i) = -x_j.$$

That is, if i has no atom at $\underline{x}_i$, then j won't want to play anything on the $(0, \underline{x}_i]$ interval. ■

Lemma 2.1.4. $\underline{x}_1 = \underline{x}_2 = 0$*. Moreover, no player will place a mass-point at* $\underline{x}_1 = \underline{x}_2 = 0$.

Proof. Suppose that $\underline{x}_i > \underline{x}_j$. By lemma 2.1.3, we know that $\underline{x}_j = 0$. Moreover, if i places no atom at $\underline{x}_i$, then j places no density on the interval $(0, \underline{x}_i]$. Thus,

$$u_i(\underline{x}_i, G_j) = G_j(0) - \underline{x}_i < \lim_{x_i \downarrow 0} u_i(\underline{x}_i, G_j) = G_j(0),$$

i.e., if i lowered the bottom of her support close to zero, she could lower her costs and not change her probabilities of winning.

If i does put an atom at $\underline{x}_i$, then by lemma 2.1.3, j would place no density on the open interval $(0, \underline{x}_i)$. We have two cases:

[3]That is, j doesn't want to bid a positive amount if she knows she is going to lose.

1. If j does not put an atom at $\underline{x}_i$, then

$$u_i(\underline{x}_i, G_j) = G_j(0) - \underline{x}_i < \lim_{x_i \downarrow 0} u_i(\underline{x}_i, G_j) = G_j(0),$$

   i.e. if i moves the point-mass at $\underline{x}_i$ close to zero she would not change her probability of winning, but would spend less on bids.

2. If j does place an atom at $\underline{x}_i$, then[4]

$$\begin{aligned} u_i(\underline{x}_i, G_j) &= G_j(0) + \frac{\mathbf{P}_j(\underline{x}_i)}{2} - \underline{x}_i \\ &< G_j(0) + \mathbf{P}_j(\underline{x}_i) - \underline{x}_i \\ &= \lim_{x_i \downarrow \underline{x}_i} u_i(\underline{x}_i, G_j), \end{aligned}$$

   if $\underline{x}_i < 1$, showing that i can do better by spreading the atom at $\underline{x}_i$ to an ε-neighborhood just above it.

   If $\underline{x}_i = 1$ instead,

$$u_j(\underline{x}_i, G_i) = \frac{\mathbf{P}_i(\underline{x}_i)}{2} - 1 < u_i(0, G_j) = 0,$$

   so that j would prefer to place that mass at 0 instead of at $\underline{x}_i$.

We have just shown that $\underline{x}_i \leq \underline{x}_j$. Of course, i and j could be reversed; as a result, $\underline{x}_1 = \underline{x}_2$.

Note that each player j will place no atom at $\underline{x}_1 = \underline{x}_2$ – otherwise, their opponent i could benefit from raising $\underline{x}_i$ slightly:

$$u_i(\underline{x}_i, G_j) = \frac{\mathbf{P}_j(\underline{x}_i)}{2} - \underline{x}_i < \lim_{x_i \downarrow \underline{x}_i} u_i(\underline{x}_i, G_j) = \mathbf{P}_j(\underline{x}_i) - \underline{x}_i.$$

By lemma 2.1.3, if player j places no atom at $\underline{x}_i = \underline{x}_j$, then $\underline{x}_i = 0$. This concludes the proof that $\underline{x}_1 = \underline{x}_2 = 0$. ■

Lemma 2.1.5. $u_1^* = u_2^* = 0$.

[4] I used "$\mathbf{P}_k(x)$" to denote the size of the atom placed at x by player k.

Proof. [5] Since no player has an atom at $\underline{x}_1 = \underline{x}_2 = 0$, $u_i^* \geq u(0, G_j) = 0$ for each $i \in I$.

Assume, by way of contradiction, that $u_i^* > 0$. Player i's expected payoffs from any set of strategies she chooses with positive probability must be exactly equal to u_i^*.

Since u is continuous around 0, we can find a small enough (but with positive measure!) interval around zero where expected payoffs would be smaller, a contradiction.

Formally, there exists $\varepsilon > 0$ such that, for all $0 < \delta < \varepsilon$, player j will play the (non-measure zero!) interval $(0, \delta)$ with positive probability.

But then, since u is continuous at 0 (from G_j being continuous at 0),

$$\lim_{\gamma \downarrow 0} u(\gamma, G_j) = u(0, G_j) = 0 < u_i^\star$$

i.e. for δ small enough, the expected payoffs from choosing in $(0, \delta)$ must be strictly less than u_i^*. This can't be, as any positive measure interval that is played in equilibrium should yield the same expected payoffs u_i^*. Thus, $u_i^* = 0$. ∎

Lemma 2.1.6. $\overline{x}_1 = \overline{x}_2 = 1$.

Proof. Suppose not. Assume $\overline{x}_i > \overline{x}_j$. Then,

$$0 = u_i^* < 1 - \overline{x}_j = \lim_{x \downarrow \overline{x}_j} u_i(x, G_j),$$

that is, i would benefit from moving mass to a point just above $\overline{x}_j$ and winning for sure, for a price $\overline{x}_j < 1$ (by lemma 2.1.2). Thus, any Nash equilibrium will be such that the top of both players' supports is at their common valuation of 1. ∎

Now that we have established the bounds of both players' supports, we begin to discuss the equilibrium distributions in earnest.

Lemma 2.1.7. *There's no mass points on the half open interval* $(0, 1]$.

[5] This proof looks complicated but the idea is very simple. Somewhat informally speaking, since 0 is in the mixed-strategy support, expected payoffs in equilibrium must equal the payoffs at 0. The difficulty is that this is not *completely* true, since 0 has measure zero, and thus technically it could have been possible for the expected payoffs from the equilibrium mixed strategies to actually be greater than the payoffs at zero. This ends up not being the case, due to u being continuous at 0.

Proof. Suppose G_i has a mass point at $x_i \in (0,1]$, i.e. it has a "jump" at x_i. For $x_i < 1$, this implies that it is worthwhile for $j \neq i$ to transfer mass from an ε-neighborhood below x_i to some δ-neighborhood above x_i, since

$$\begin{aligned}\lim_{x_j \uparrow x_i} u_j(x_j, G_i) &= G_i(x_i) - \mathbf{P}_i(x_i) - x_i \\ &< G_i(x_i) - x_i \\ &= \lim_{x_j \downarrow x_i} u_j(x_i, G_i).\end{aligned}$$

For $x_i = 1$, it pays for j to transfer mass from an ε-neighborhood below x_i to zero, since:

$$\lim_{x_j \uparrow 1} u_j(x_j, G_i) = 1 - \mathbf{P}_i(x_i) - 1 < u_j(0, G_j) = 0.$$

Either way, there would be an ε-neighborhood below x_i in which j would put no mass. But then it can't be an equilibrium strategy for i to put mass at x_i. In symbols,

$$\begin{aligned}&\exists \varepsilon > 0, \quad \forall \delta \in (0, \varepsilon), \\ &u_i(x_i, G_j) = G_j(x_i - \varepsilon) - x_i \\ &\qquad\qquad < u_i(x_i - \delta, G_j) \\ &\qquad\qquad = G_j(x_i - \varepsilon) - x_i + \delta,\end{aligned}$$

i.e. i would benefit from moving mass from x_i to $x_i - \delta$ in the interval where j is not bidding.

This shows that any mass-point in the $(0,1]$ interval is incompatible with both players choosing best-responses to their opponents' actions. We have previously shown that there will be no atoms at 0 either. Thus, a Nash equilibrium of the all-pay auction will not have any mass-points. ■

Lemma 2.1.8. *The two players randomize continuously with full* $[0,1]$ *support.*

Proof. Suppose that there is an interval $(x', x'') \subset [0,1]$ where player i places no probability mass. Pick one such interval where, for each $\varepsilon > 0$, $G_i(x'' + \varepsilon) - G_i(x'') > 0$, i.e. x'' is the "largest" point satisfying[6] $G(x') = G(x'')$. This

[6]Such a x'' exists since there are no mass-points in i's distribution, implying G_i is continuous. Thus, if $x'' = \sup\{x \in [0,1] : G_i(x) \leq G_i(x')\}$, then $G_i(x'') = G_i(x')$.

is just to make sure there's some probability mass in the interval just above x''.

We have that j can't have any density on (x', x'') either, or she would rather transfer it all to x': after all, for all $x \in (x', x'')$,

$$u_j(x, G_i) = G_i(x') - x < G_i(x') - x' = u_j(x', G_i).$$

Since j is not playing anything on (x', x''),

$$\lim_{x \downarrow x''} u_i(x, G_j) = G_j(x') - x'' < u_i(x', G_j) = G(x') - x'.$$

i.e. anything too close to x'' from above is worse than picking x', and so i would benefit from transferring some mass from the ε-neighborhood of x'' to x'.

This shows that a "gap" in the support means one of the players can't possibly be choosing a best-response to their opponent's actions. Therefore, in any Nash equilibrium the two players will randomize continuously on the full $[0, 1]$ support. ■

Now we are ready to finish proving the Theorem.

If $x \in \operatorname{supp} G_i = [0, 1]$, then

$$u_i^* = 0 = G_j(x) - x$$

that is, $G_j(x) = x$.

Of course, the same argument implies $G_i(x) = x$ in $[0, 1]$ as well, which concludes the proof. ■

References

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